Multi-Spaces and Many Worlds from Conservation Laws

Giuseppe Basini* and Salvatore Capozziello†

*Laboratori Nazionali di Frascati, INFN, Via E. Fermi C.P. 13, I-0044 Frascati (Roma) Italy

‡Dipartimento di Scienze Fisiche, Università di Napoli "Federico II", and INFN, Sez. di Napoli Complesso Universitario di Monte S. Angelo, Via Cinthia, Edificio N-I-80126 Napoli, Italy

Corresponding author, e-mail: capozziello@na.infn.it

Many Worlds interpretation of Quantum Mechanics can be related to a General Conservation Principle in the framework of the so called Open Quantum Relativity. Specifically, conservation laws in phase space of physical systems (e. g. minisuperspace) give rise to natural selection rules by which it is possible to discriminate among physical and unphysical solutions which, in the specific case of Quantum Cosmology, can be interpreted as physical and unphysical universes. We work out several examples by which the role of conservation laws is prominent in achieving the solutions and their interpretation.

1 Introduction

The issue to achieve a unified field theory cannot overcome to take into account the role and meaning of conservation laws and dynamical symmetries which have always had a fundamental role in physics. From a mathematical viewpoint, their existence allows to "reduce" the dynamics and then to obtain first integrals of motion, which often allow the exact solution of the problem of motion. Noether theorem is a prominent result in this sense, since it establishes a deep link between conservation laws and symmetries. Moreover, conservation laws can play a deep role in the definition of physical theories and, in particular, to define space-times which are of *physical interest*. The underlying philosophy is the fact that the violation of conservation laws (and then the symmetry breaking) could be nothing else but an artificial tool introduced in contemporary physics in order to solve phenomenologically some puzzles and problems, while effective conservation laws are never violated [1]. The absolute validity of conservation laws, instead, allows the solution of a wide variety of phenomena ranging from entanglement of physical systems [2], to the rotation curves of spiral galaxies [4]. Such results do not come from some *a priori* request of the theory, but is derived from the existence of a *General Conservation Law* (in higher dimensional space-time) where no violation is allowed [5]. This approach naturally leads to a dynamical unification scheme (the so called Open Quantum Relativity [1]) which can be, as a minimal extension, formulated in 5D [6]. In this context, it is worth stressing the deep relations among symmetries and first integrals of motion, conservation laws with the number and dimensionality of configuration spaces. In fact, phenomena, which in standard physics appear as due to symmetry breakings can be encompassed in a multi-space formulation as previously shown by Smarandache [7, 8]. On the other hand, the need of a multi-space formulation of the theory gives rise to a direct application of the "Many Worlds" Interpretation of Quantum Mechanics [9, 10], in the sense that multi-spaces are nothing else but many worlds in the framework of Quantum Cosmology [11]. This is the argument of this paper: we want to show that configuration spaces derived from the request of integrability of the dynamical systems (and then from the presence of conservation laws) are physical universes, (i. e. observable universes) where cosmological parameters can be observed. On the other hand, if conservation laws are not present, in universes which come out in a Many Worlds interpretation are "unphysical" that is, it is not possible to label them by a set of observable cosmological parameters (technically they are "instanton-solutions"). In Sect. 2, we develop mathematical considerations on conservation laws showing how the presence of symmetries allows the integration of the dynamical systems, which means that the phase-space (and general solution) can be "split" in a multi-space of "integrated" components. Sect. 3 is devoted to the discussion of Many Worlds interpretation of Quantum Cosmology and, in particular, to the fact that multi-spaces related to the phase-space of conservation laws can be interpreted as "minisuperspaces" thanks to the Hartle criterion. Many Worlds-solutions from conservation laws are obtained in Sect. 4 by integrating the Wheeler-DeWitt (WDW) equation of Quantum Cosmology. Conclusions are drawn in Sect. 5.

2 Conservation laws and multi-spaces

Before considering multi-spaces and how they can be interpreted as the Many Worlds of Quantum Cosmology, let us discuss the reduction problem of dynamics connected symmetries and conservation laws. Our issue is to show that the total phase-space of a given dynamical system can be split in many subspaces, each of them related to a specific conserved quantity. As a general remark, it is possible to show that if the Lie derivative of a given geometric quantity (e. g. vector, tensor, differential form) is zero, such a quantity is conserved. This property is *covariant* and specifies the number of dimensions and the nature of configuration space (and then of the phase-space) where the given dynamical system is defined. Furthermore, the existence of conserved quantities always implies a *reduction* of dynamics which means that the order of equations of motion is reduced thanks to the existence of first integrals. Before considering specific systems, let us remind some properties of the Lie derivative and how conservation laws are related to it. Let L_X be the Lie derivative

$$(L_X\omega)\,\xi = \frac{d}{dt}\,\omega\,(g^t_*\xi)\,, \tag{1}$$

where ω is a differential form of $\mathcal{R}^n$ defined on the vector field ξ, g^t_* is the differential of the phase flux $\{g_t\}$ given by the vector field X on a differential manifold $\mathcal{M}$. The discussion can be specified by considering a Lagrangian $\mathcal{L}$ which is a function defined on the tangent space of configurations $T\mathcal{Q}\equiv\{q_i,\dot{q}_i\}$, that is $\mathcal{L}:T\mathcal{Q}\rightarrow\Re$. In this case, the vector field X is

$$X = \alpha^i(q)\frac{\partial}{\partial q^i} + \dot{\alpha}^i(q)\,\frac{\partial}{\partial \dot{q}^i}\,, \tag{2}$$

where the dot denotes the derivative with respect to t, and we have

$$L_X\mathcal{L} = X\mathcal{L} = \alpha^i(q)\,\frac{\partial\mathcal{L}}{\partial q^i} + \dot{\alpha}^i(q)\,\frac{\partial\mathcal{L}}{\partial \dot{q}^i}\,. \tag{3}$$

It is important to note that t is simply a parameter which specifies the evolution of the system. The condition

$$L_X\mathcal{L} = 0 \tag{4}$$

implies that the phase flux is conserved along X: this means that a constant of motion exists for $\mathcal{L}$ and a conservation law is associated to the vector X. In fact, by taking into account the Euler-Lagrange equations, it is easy to show that

$$\frac{d}{dt}\left(\alpha^i\,\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right) = L_X\mathcal{L}\,. \tag{5}$$

If (4) holds, the relation $\Sigma_0 = \alpha^i\,\frac{\partial\mathcal{L}}{\partial\dot{q}^i}$ identifies a constant of motion. Alternatively, using a generalized differential for the Lagrangian $\mathcal{L}$, the Cartan one–form, $\theta_{\mathcal{L}} \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\,dq^i$ and defining the inner derivative $i_X\theta_{\mathcal{L}} = \langle\theta_{\mathcal{L}}, X\rangle$, we get

$$i_X\theta_{\mathcal{L}} = \Sigma_0 \tag{6}$$

if, again, condition (4) holds. This representation identifies cyclic variables. Using a point transformation on vector field (2), it is possible to get

$$\tilde{X} = (i_X dQ^k)\,\frac{\partial}{\partial Q^k} + \left[\frac{d}{dt}(i_X dQ^k)\right]\frac{\partial}{\partial\dot{Q}^k}\,. \tag{7}$$

From now on, Lagrangians and vector fields transformed by the non–degenerate transformation

$$Q^i = Q^i(q)\,,\qquad \dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j}\,\dot{q}^j \tag{8}$$

will be denoted by a tilde. If X is a symmetry for the Lagrangian $\mathcal{L}$, also $\tilde{X}$ is a symmetry for the Lagrangian $\tilde{\mathcal{L}}$ giving rise to a conserved quantity, thus it is always possible to choose a coordinate transformation so that

$$i_X dQ^1 = 1\,,\qquad i_X dQ^i = 0\,,\qquad i\neq 1\,, \tag{9}$$

and then

$$\tilde{X} = \frac{\partial}{\partial Q^1}\,,\qquad \frac{\partial\tilde{\mathcal{L}}}{\partial Q^1} = 0\,. \tag{10}$$

It is evident that Q^1 is a cyclic coordinate because dynamics can be reduced. Specifically, the "reduction" is connected to the existence of the second of (10). However, the change of coordinates is not unique and an opportune choice of coordinates is always important. Furthermore, it is possible that more symmetries are existent. In this case more cyclic variables must exist. In general, a reduction procedure by cyclic coordinates can be achieved in three steps: (i) we choose a symmetry and obtain new coordinates as above and after this first reduction, we get a new Lagrangian $\tilde{\mathcal{L}}$ with a cyclic coordinate; (ii) we search for new symmetries in this new space and iterate the reduction technique until it is possible; (iii) the process stops if we select a pure kinetic Lagrangian where all coordinates are cyclic. In such a case, the dynamical system is *completely integrable* and integration can be achieved along every coordinate of configuration space (or every generalized coordinate-conjugate momentum couple of phase space). In this case, the total phase-space is split in subspaces, each one *labelled* by a conserved quantity. Technically, every symmetry selects a constant conjugate momentum since, by the Euler–Lagrange equations we get

$$\frac{\partial\tilde{\mathcal{L}}}{\partial Q^i} = 0 \Longleftrightarrow \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{Q}^i} = \Sigma_i\,, \tag{11}$$

and the existence of a constant conjugate momentum means that a cyclic variable (a symmetry) exists.

However, The Lagrangian $\mathcal{L} = \mathcal{L}\,(q^i,\dot{q}^j)$ has to be non-degenerate, which means that the Hessian determinant has to be non-zero.

From the Lagrangian formalism, we can pass to the Hamiltonian one through the Legendre transformation

$$\mathcal{H} = \pi_j\dot{q}^j - \mathcal{L}\,(q^j,\dot{q}^j)\,,\qquad \pi_j = \frac{\partial\mathcal{L}}{\partial\dot{q}^j}\,, \tag{12}$$

defining, respectively, the Hamiltonian function and the conjugate momenta. In the Hamiltonian formalism, the conservation laws are obtained when $\left[\Sigma_j,\mathcal{H}\right] = 0$, $1\leqslant j\leqslant m$ This is the relation for conserved momenta and, in order to obtain a symmetry, the Hamilton function has to satisfy the relation

$L_\Gamma \mathcal{H} = 0$, where the vector Γ is defined by

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i}\,. \tag{13}$$

Let us now go to the specific formalism of Quantum Mechanics which we will use for the following Quantum Cosmology considerations. By the Dirac canonical quantization procedure, we have

$$\pi_j \longrightarrow \hat{\pi}_j = -i\partial_j\,, \qquad \mathcal{H} \longrightarrow \hat{\mathcal{H}}(q^j, -i\partial_{q^j})\,. \tag{14}$$

If $|\Psi\rangle$ is a *state* of the system (i. e. its wave function), dynamics is given by the Schrödinger eigenvalue equation

$$\hat{\mathcal{H}}|\Psi\rangle = E|\Psi\rangle\,, \tag{15}$$

where, obviously, the whole wave-function is given by $|\phi(t,x)\rangle = e^{iEt/\hbar}|\Psi\rangle$. If a symmetry exists, the reduction procedure outlined above can be applied and then, from (11) and (12), we get

$$\begin{aligned} \pi_1 &\equiv \frac{\partial \mathcal{L}}{\partial \dot{Q}^1} = i_{X_1}\theta_{\mathcal{L}} \;=\; \Sigma_1\,, \\ \pi_2 &\equiv \frac{\partial \mathcal{L}}{\partial \dot{Q}^2} = i_{X_2}\theta_{\mathcal{L}} \;=\; \Sigma_2\,, \\ &\quad\;\;\dots \quad \dots \qquad \dots\,, \end{aligned} \tag{16}$$

depending on the number of symmetry vectors. After Dirac quantization, we get

$$-i\partial_1|\Psi\rangle = \Sigma_1|\Psi\rangle,\;\; -i\partial_2|\Psi\rangle = \Sigma_2|\Psi\rangle,\;\; \dots \tag{17}$$

which are nothing else but translations along the Q^j axis singled out by the corresponding symmetry. Eqs. (17) can be immediately integrated and, being Σ_j real constants, we obtain oscillatory behaviors for $|\Psi\rangle$ in the directions of symmetries, i. e.

$$|\Psi\rangle = \sum_{j=1}^{m} e^{i\Sigma_j Q^j}|\chi(Q^l)\rangle\,, \qquad m < l \leqslant n\,, \tag{18}$$

where m is the number of symmetries, l are the directions where symmetries do not exist and n is the number of dimensions of configuration space. Vice-versa, dynamics given by (15) can be reduced by (17) if, and only if, it is possible to define constant conjugate momenta as in (16), *i.e.* oscillatory behaviors of a subset of solutions $|\Psi\rangle$ exist as a consequence of the fact that symmetries are present in the dynamics. The m symmetries give first integrals of motion. In one and two–dimensional configuration spaces, the existence of a symmetry allows the complete solution of the problem. Therefore, if $m = n$, the problem is completely solvable and a symmetry exists for every variable of configuration space. The reduction procedure of dynamics, connected to the existence of symmetries, allows to select a subset of the general solution of equations of motion, where oscillatory behaviors of the wave functions are found. In other words, symmetries select exact solutions and reduce dynamics. In these cases, the general solution of a dynamical system can be split in a combination of functions each of them depending on a given variable. As a corollary, a Lagrangian (or a Hamiltonian) where only kinetic terms are present gives always rise to a full integrable dynamics. The total phase-space $\mathcal{M}$ of the system, thanks to conservation laws, can be split in the tensor product of phase-spaces (multi-spaces) assigned by conserved momenta, i. e. $\{q_i, \pi_i\} \to \{Q_i, \Sigma_i\}$, and then $\mathcal{M} = \Pi_{i=1}^{n}\{Q_i, \bigotimes \Sigma_i\}$. As we will see, this feature is relevant in minisuperspace Quantum Cosmology.

3 The "many-worlds" interpretation of Quantum Mechanics and the role of conservation laws

The above considerations acquire a fundamental role in Minisuperspace Quantum Cosmology since, as we will see, Conservation Laws give rise to an approach by which it is possible to "select" physical universes. Quantum Cosmology is one of the results of the efforts of last thirty years directed to the quantization of gravity [12]. The aim has been to obtain a scheme in which gravity is treated on the same ground of the other interaction of Nature. Such an approach (not a coherent theory yet) is the *canonical quantization of gravity*. In order to test the theoretical results, Planck's scales, which cannot be reached by the current physics, have to be considered, so the cosmology is the most reasonable area for the application of the observable predictions of quantum gravity. More properly, Quantum Cosmology is the quantization of dynamical systems which are "*universes*". In this context, supposed the Universe as a whole (the ensemble of all the possible universes), it has a quantum mechanical nature and that an observable universe is only a limit concept valid in particular regions of a manifold (*superspace*) composed by all the possible space-like 3-geometries and local configurations of the matter fields. The task of Quantum Cosmology is to relate all the measurable quantities of the observable universe* to the assigned boundary conditions for a wave function in the superspace. This wave function has to be connected to the probability to obtain typical universes (even if, in the standard approach, it is not a proper probability amplitude since a Hilbert space does not exist in the canonical formulation of quantum gravity) [11]. Quantum Cosmology has to solve, in principle, the problem of the initial conditions of the standard cosmology: *i.e.* it should explain the observed universe, specifying the physical meaning of the boundary conditions of the superspace wave function. In other words, the main issue of quantum cosmology is to search for boundary conditions in agreement with the

*An operative definition of "observable universe" could be a universe where cosmological parameters as the Hubble one H_0, the deceleration parameter q_0, the density parameters Ω_M, Ω_Λ, Ω_k and the age t_0 can be inferred by observations [3].

astronomical observations and these conditions have to be contained in the wave function of the universe $|\Psi\rangle$. The dynamical behavior of $|\Psi\rangle$ in the superspace is described by the Wheeler-DeWitt (WDW) equation [12] that is a second order functional differential equation hard to handle, because it has infinite degrees of freedom. Usually attention has been concentrated on finite dimensional models in which the metrics and the matter fields are restricted to particular forms (*minisuperspace models*), like homogeneous and isotropic spacetimes. With these choices, the WDW equation becomes a second order partial differential equation which, possibly, can be exactly integrated. However, by definition, there is no *rest outside of the Universe* in cosmology, so that boundary conditions must be considered as a *fundamental law of physics* [11]. Moreover, not only the conceptual difficulties, but also the mathematical ones, make Quantum Cosmology hard to handle. For example, the superspace of geometrodynamics [13] has infinite degrees of freedom so that it is technically impossible to integrate the full infinite dimensional WDW equation. Besides, a Hilbert space of states describing the universe is not available [12]. Finally, it is not well established how to interpret the solutions of WDW equation in the framework of probability theory. Despite these still unsolved shortcomings, several positive results have been obtained and Quantum Cosmology has become a sort of *paradigm* in theoretical physics researches. For example the infinite-dimensional superspace can be restricted to opportune finite-dimensional configuration spaces called *minisuperspaces*. In this case, the above mathematical difficulties can be avoided and the WDW equation can be integrated. The so called *no boundary condition* by Harte and Hawking [14] and the *tunneling from nothing* by Vilenkin [15] give reasonable laws for initial conditions from which our observable universe could be started. The *Hartle criterion* [11] is an interpretative scheme for the solutions of the WDW equation. Hartle proposed to look for peaks of the wave function of the universe: if it is strongly peaked, we have correlations among the geometrical and matter degrees of freedom; if it is not peaked, correlations are lost. In the first case, the emergence of classical relativistic trajectories (*i.e.* universes) is expected. The analogy to the quantum mechanics is immediate. If we have a potential barrier and a wave function, solution of the Schrödinger equation, we have an oscillatory regime upon and outside the barrier while we have a decreasing exponential behavior under the barrier. The situation is analogous in Quantum Cosmology: now potential barrier has to be replaced by the superpotential $U(h^{ij},\varphi)$, where h^{ij} are the components of the three–metric of geometrodynamics and φ is a generic scalar field describing the matter content. More precisely, the wave function of the universe can be written as

$$\Psi\left[h_{ij}(x),\phi(x)\right]\sim e^{im_P^2 S}\,, \tag{19}$$

where m_p is the Planck mass and

$$S\equiv S_0+m_P^{-2}S_1+\mathrm{O}(m_P^{-4}) \tag{20}$$

is the action which can be expanded. We have to note that there is no normalization factor due to the lack of a probabilistic interpretative full scheme. Inserting S into the WDW equation and equating similar power terms of m_p, one obtains the Hamilton-Jacobi equation for S_0. Similarly, one gets equations for $S_1, S_2\ldots$, which can be solved considering results of previous orders giving rise to the higher order perturbation theory. We need only S_0 to recover the semi-classical limit of Quantum Cosmology [10]. If S_0 is a real number, we get oscillating WKB modes and the Hartle criterion is recovered since $|\Psi\rangle$ is peaked on a phase-space region defined by

$$\pi_{ij}=m_P^2\,\frac{\delta S_0}{\delta h^{ij}}\,,\qquad \pi_\varphi=m_P^2\,\frac{\delta S_0}{\delta\varphi}\,, \tag{21}$$

where π_{ij} and π_φ are classical momenta conjugates to h^{ij} and φ. It is worth stressing, at this point, that such a momenta are nothing else but Conservation Laws. The semi-classical region of superspace, where Ψ has an oscillating structure, is the Lorentz one otherwise it is Euclidean*. In the latter case, we have $S=iI$ and

$$\Psi\sim e^{-m_P^2 I}\,, \tag{22}$$

where I is the action for the Euclidean solutions of classical field equations (*istantons*). This scheme, at least at a semi-classical level, solves the problem of initial conditions. Given an action S_0, Eqs. (21) imply n free parameters (one for each dimension of the configuration space $\mathcal{Q}\equiv\{h^{ij},\varphi\}$) and then n first integrals of motion exactly as in the scheme proposed in the previous section. However the general solution of the field equations involves $2n-1$ parameters (one for each Hamilton equation of motion except the energy constraint). Consequently, the wave function is peaked on a subset of the general solution. In this sense, the boundary conditions on the wave function imply initial conditions for the classical solutions. In other words, the issue is searching for some general method by which selecting such constants of motion related to the emergence of classical trajectories without arbitrarily choosing regions of the phase-space where momenta are conserved. In this sense, there is a deep connection between the conservation laws and the structure of the wave function of the universe. Using the results of the previous section (see Eq. 18), the oscillatory regime, and then the correlation among the variables in the framework of the Hartle criterion, is guaranteed only if conservation laws are present into dynamics. In this context, if conservation laws are absolutely valid, the above *reduction procedure* gives rise to subsets of the infinite dimensional general solution of

*It is important to note that we are using both symbols $|\Psi\rangle$ and Ψ depending on the interpretation which we want to give to the wave function. In the first case, the wave function is considered a "quantum-state", in the second one, it has a semi-classical interpretation.

the WDW equation where oscillating behaviors are recovered. Viceversa, the Hartle criterion is always connected to the presence of a conservation law and then to the emergence of classical trajectories which are *observable universes* where cosmological observations are possible. Then the above result can be given in the following way:

> *In minisuperspace quantum cosmology, the existence of conservation laws yields a reduction procedure of dynamics which allows to find out oscillatory behaviors for the general solution of WDW equation. Viceversa, if a subset of the solution of WDW equation has an oscillatory behavior, conserved momenta have to exist and conservation laws are present. If a conservation law exists for every configuration variable, the dynamical system is completely integrable and the general solution of WDW equation is a superposition of oscillatory behaviors. In other words, conservation laws allow and select observable universes.*

On the other hand, if conservation laws are not valid the WDW multi-space solution give rise to non-observable universes (instanton solutions).

4 Many worlds from conservation laws

In order to give concrete examples of the above results, we can show how, given a generic theory of gravity, it is possible to work out minisuperspace cosmological models where observable universes (classical trajectories) are obtained thanks to the existence of conserved quantities. We shall take into account the most general action in which gravity is nonminimally coupled to a scalar field:

$$\mathcal{A}=\int_{\mathcal{M}} d^4x\sqrt{-g}\left[F(\varphi)R+\frac{1}{2}\,g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu}-V(\varphi)+\mathcal{L}_m\right] \quad (23)$$

where the form and the role of $V(\varphi)$ are still general and $\mathcal{L}_m$ represents the standard fluid matter content of the theory. This effective action comes out in the framework of the Open Quantum Relativity [1, 6] a dynamical theory in which, asking for a General Conservation Principle [5], the unification of different interactions is achieved and several shortcomings of modern physics are overcome (see [1] and references therein). The state equation of fluid matter is $p=(\gamma-1)\,\rho$ and $1\leqslant\gamma\leqslant 2$ where p and ρ are, respectively, the ordinary pressure and density. Now we have all the ingredients to develop a scalar-tensor gravity quantum cosmology. Using the transformations:

$$\varphi=e^{-\varphi}, \quad F(\varphi)=\frac{1}{8}\,e^{-2\varphi}, \quad V(\varphi)=U(\varphi)\,e^{-2\varphi}, \quad (24)$$

the action (23) can be recast in the form

$$\mathcal{A}=\int d^4x\sqrt{-g}\,\big\{\exp\left[-2\varphi\right]\left[R+4\,g^{\mu\nu}\varphi_{;\,\mu}\varphi_{;\,\nu}+ - U(\varphi)\right]+\mathcal{L}_m\big\}\,, \quad (25)$$

always using Planck units $8\pi G=c=1$. Let us now take into account a Friedman, Robertson, Walker (FRW) metric $ds^2=dt^2-a^2(t)d\Omega_3^2$, where $d\Omega_3^2$ is the 3–dimensional element of the spacelike manifold. With this assumption, the configuration space is $\mathcal{Q}\equiv\{a,\varphi\}$ and the tangent space is $T\mathcal{Q}\equiv\{a,\dot{a},\varphi,\dot{\varphi}\}$. This is our minisuperspace. Clearly $p=p(a)$ and $\rho=\rho(a)$. Substituting the FRW metric and integrating by parts, the Lagrangian (25) becomes point-like, that is:

$$\mathcal{L}=\frac{1}{8}\,a^3e^{-2\varphi}\left[6\left(\frac{\dot{a}}{a}\right)^2-12\,\dot{\varphi}\left(\frac{\dot{a}}{a}\right)-6\,\frac{k}{a^2}+ +4\,\dot{\varphi}^2-8\,U(\varphi)\right]+a^3\mathcal{L}_m\,. \quad (26)$$

At this point, it is worth noting that the scale–factor duality symmetry arises if the transformation of the scale factor of a homogeneous and isotropic space-time metric, $a(t)\to a^{-1}(-t)$, leaves the model invariant, taking into account also the form of the potential U.

Provided the transformations

$$\psi=\varphi-\frac{3}{2}\ln a\,, \qquad Z=\ln a\,, \quad (27)$$

the Lagrangian (26) becomes:

$$\mathcal{L}=e^{-2\psi}\left[4\dot{\psi}^2-3\dot{Z}^2-6ke^{-2Z}-8W\right]+De^{3(1-\gamma)Z} \quad (28)$$

where the potential $W(\psi,Z)$, thanks to the transformations (27), is depending on both the variables of the minisuperspace. In the new variables, the duality invariance has become a parity invariance since Z and $-Z$ are both solutions of dynamics. The emergence of this feature is related to the presence of nonminimal coupling; it allows the fact that several solutions can be extended for $t\to-\infty$ without singularities [3]. Another important consideration is connected to the role of perfect fluid matter. It introduces two further parameters which are D (related to the bulk of matter) and γ (related to the type of matter which can be *e.g.* radiation $\gamma=4/3$ or dust $\gamma=1$). We shall see below that they directly determine the form of cosmological solutions. Two general forms of potential W preserving the duality symmetry

$$W(Z,\psi)=\frac{D}{4}\,e^{-3\gamma Z}e^{2\psi}\,, \qquad W(Z,\psi)=\Lambda\,, \quad (29)$$

where $\Lambda=$const. These are all the ingredient we need in order to construct our minisuperspace quantum cosmology. Let us start with a simple but extremely didactic example of the above effective action (25) which is

$$\mathcal{A}=\int d^4x\sqrt{-g}\,e^{-2\varphi}\left[R+4(\partial\varphi)^2-\Lambda\right], \quad (30)$$

where $D=k=0$ and $W=\Lambda$. This example is useful to show, as we shall see below, the way in which the full theory works. The Lagrangian (28) becomes

$$\mathcal{L} = -3e^{-2\psi}\dot{Z}^2 + 4\dot{\psi}^2 e^{-2\psi} - 2\Lambda e^{-2\psi}, \tag{31}$$

that is cyclic in Z. Due to the considerations in previous section, we have to derive a conserved quantity, relatively to the variable Z, and then an oscillatory behavior for the wave function of the universe Ψ. The Legendre transformation for the conjugate momenta gives

$$\pi_z = \frac{\partial\mathcal{L}}{\partial\dot{Z}} = -6\dot{Z}e^{-2\psi}, \quad \pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = 8\dot{\psi}e^{-2\psi}, \tag{32}$$

and the Hamitonian is $\mathcal{H} = \pi_z\dot{Z} + \pi_\psi\dot{\psi} - \mathcal{L}$. From Dirac canonical quantization rules, it is possible to write $\pi_z \to -i\partial_z$ and $\pi_\psi \to -i\partial_\psi$, and then the WDW equation is

$$\left[\frac{1}{12}\,\partial_z^2 - \frac{1}{16}\,\partial_\psi^2 + 2\Lambda e^{-4\psi}\right]\Psi(Z,\psi) = 0\,, \tag{33}$$

where a simple factor ordering choice is done [11]. This is a second order partial differential equation which can be solved by separation of variables $\Psi(Z,\psi) = A(Z)\,B(\psi)$ from which Eq. (33) can be split into two ordinary differential equations

$$\frac{d^2B(\psi)}{d\psi^2} - \left(32\Lambda e^{-4\psi} + 16E\right)B(\psi) = 0\,, \tag{34}$$

$$\frac{d^2A(z)}{dz^2} = 12EA(z)\,, \tag{35}$$

where E is a arbitrary constant. For $E > 0$, the general solution of the WDW equation is

$$\begin{aligned}\Psi(Z,\psi) &\propto \exp\left(\pm\sqrt{\frac{3}{2}}Z\right)\times\\ &\times\left[c_0 \mp \frac{1}{8\sqrt{2\Lambda}}\exp\left(\pm4\sqrt{2\Lambda}\,e^{-2\psi}\right)\right]\times\\ &\times\exp\left[\psi \mp 2\sqrt{2\Lambda}\,e^{-2\psi}\right].\end{aligned} \tag{36}$$

For $E < 0$, Eq. (35) is a harmonic oscillator whose solutions are $A(Z) \propto \pm\sin(mZ)$ (we have put $|E| = m^2$). In this case the momentum $\pi_z = m$ is a constant of motion. Eq. (34) is solvable in terms of modified Bessel functions and the general solution of Eq. (33) is

$$\Psi(Z,\psi) \propto \pm\sin(mZ)\,K_{\frac{im}{2\Lambda}}\left(\sqrt{2\Lambda}\,e^{-2\psi}\right); \tag{37}$$

with an evident oscillatory behavior. Finally, in the case $E = 0$, the solution is

$$\Psi(Z,\psi) \propto ZK_0\left(\sqrt{2\Lambda}\,e^{-2\psi}\right), \tag{38}$$

where K_0 is the modified Bessel function of zero order. The absence of a positive defined scalar product in the superspace prevents the existence of a Hilbert space for the states of the WDW equation; *i. e.* we cannot apply the full probability interpretation to the squared modulus of the wave function of the universe. This is the reason why we have to omit the normalization constants in front of the solutions (36), (37), (38). Various suggestions have been given in literature to interpret Ψ [11], although starting from different points of view, all these different interpretations arrive to the conclusion that, at least in the semiclassical limit, a notion of measure can be introduced considering $|\Psi|^2$. As we said above, the strong peaks of $|\Psi|^2$ (oscillatory behaviors) indicate classical correlations among the dynamical variables, whereas weak variations of $|\Psi|^2$ mean the absence of correlations [11]. In fact the presence of strong amplitude peaks of the wave function seems to be the common indicator of where the classical (in principle observable) universes enucleates in its configuration space. The classical limit of quantum cosmology can be recovered in the oscillation regime with great phase values of Ψ: in this region the wave function is strongly peaked on first integrals of motion related to conservation laws. In the case presented here, the solutions (36), (37), (38) give information on the nature and the properties of classical cosmological behavior: for the *vacuum state*, $E = 0$, we have

$$\Psi \sim \ln a\sqrt{\pi}\,e^{\psi}\exp\left(-\sqrt{2\Lambda}\,e^{-2\psi}\right) \to 0\,, \tag{39}$$

for $\psi \to -\infty$ and

$$\Psi \sim 2\psi\ln a\,, \tag{40}$$

for $\psi \to +\infty$. So $|\Psi|^2$ is exponentially small for $\psi \leqslant 0$, while it increases for great ψ. This fact tells us that is most probable a realization of a classical universe for great field configurations (for example see the prescriptions for chaotic inflation where the scalar field has to start with a mass of a few Planck masses [16]). Another feature which emerges from (36) and (37) is the following: as $Z = \ln a$, Ψ can be considered a superposition of states $\Psi(a)$ with states $\Psi(a^{-1})$, that is the wave function of the universe (and also the WDW equation) contains the scale factor duality. Furthermore, using the first integrals of motion (*i.e* the canonical momenta related to conservation laws), we get the classical solutions

$$a(t) = a_0\left[\frac{\cos\lambda\tau + \sin\lambda\tau}{\cos\lambda\tau - \sin\lambda\tau}\right]^{\pm\sqrt{3}/3}, \tag{41}$$

$$\begin{aligned}\varphi(t) = \frac{1}{4}\ln\left[\frac{\lambda^2}{k\cos^2 2\lambda\tau}\right] &\pm\\ \pm\frac{\sqrt{3}}{2}\ln&\left|\frac{\cos\lambda\tau + \sin\lambda\tau}{\cos\lambda\tau - \sin\lambda\tau}\right| + \varphi_0\,,\end{aligned} \tag{42}$$

and

$$a(t) = a_0\exp\left\{\mp\frac{1}{\sqrt{6}}\arctan\left[\frac{1 - 2e^{4\lambda\tau}}{2e^{2\lambda\tau}\sqrt{1 - e^{4\lambda\tau}}}\right]\right\}, \tag{43}$$

Λ	k	D	γ	*Solution*
$\neq 0$	0	$\neq 0$	1	CT
0	0	$\neq 0$	1	CT
0	± 1	$\neq 0$	4/3	I
$\neq 0$	0	0	$\forall\gamma$	CT
0	$k > 0$	$6k$	$\forall\gamma$	CT and I

Table 1: Main features of the solutions of WDW equation, Classical Trajectories (CT) and Instantons (I), for different values of parameter Λ, k, D, γ.

$$\varphi(t) = \frac{1}{4}\ln\left[\frac{2\lambda^2 e^{4\lambda\tau}}{k\,(1-e^{4\lambda\tau})}\right] \mp \mp \frac{1}{\sqrt{6}}\arctan\left[\frac{1-2e^{4\lambda\tau}}{2e^{2\lambda\tau}\sqrt{1-e^{4\lambda\tau}}}\right] + \varphi_0\,, \tag{44}$$

where $\tau = \pm t$, k is an integration constant and $\lambda^2 = \Lambda/2$. In (41), (42), we have $\Lambda > 0$, in (43), (44) $\Lambda < 0$. These "universes" are "observable" since, starting from these solutions, it is easy to construct all the cosmological parameters H_0, q_0, Ω_Λ, Ω_M and t_0. It is worth stressing that such solutions are found only if conservation laws exists. It is remarkable that the scalar factor duality emerges also for the wave function of the universe in a quantum cosmology context: that is the solutions for a have their dual counterpart a^{-1} in the quantum state described by Ψ. This fact, in the philosophy of quantum cosmology, allows to fix a law for the initial conditions (*e. g.* Vilenkin tunneling from nothing or Hartle-Hawking no-boundary conditions [11]) in which the duality is a property of the configuration space where our classical universe enucleates. This fact gives rise to cosmological solutions which can be consistently defined for $t \to \pm\infty$.

The approach can be directly extended to the Lagrangian (28), from which, by a Legendre transformation and a canonical quantization, we get the WDW equation

$$\left[\frac{1}{2}\,\partial_z^2 - \frac{1}{8}\,\partial_\psi^2 + 3ke^{-2Z-4\psi} + + 4We^{-4\psi} - De^{3(1-\gamma)Z-2\psi}\right]\Psi(Z,\psi) = 0\,, \tag{45}$$

whose solutions can be classified by the potential parameter Λ, the spatial curvature k, the bulk of matter D, and the adiabatic index γ. In the following Table, we give the main features of WDW solutions.

5 Discussion and conclusions

In this paper, we have shown that the reduction procedure of dynamics, related to conservation laws, can give rise to a splitting of the phase-space of a physical system, by which it is possible to achieve the complete solution of dynamics. This result can be applied to Quantum Cosmology, leading to the result that physical many worlds can be related to integrable multi-spaces of the above splitting. From a mathematical viewpoint, the above statement deserves some further discussion. As a first remark the general solution (18) can be interpreted as a superposition of particular solutions (the components in different directions) which result more *solved* (*i.e.* separated in every direction of configuration space) if more symmetries exist. Starting from such a consideration, as a consequence, we can establish a sort of *degree of solvability*, among the components of a given physical system, connected to the number of symmetries: (i) a system is *completely* solvable and separable if a symmetry exists for *every* direction of configuration space (in this case, the system is fully integrable and the relations among its parts can be exactly obtained); (ii) a system is *partially* solved and separated if a symmetry exists for *some* directions of configuration space (in this case, it is not always possible to get a general solution); (iii) a system is *not* separated at all and *no* symmetry exists, i. e. a necessary and sufficient condition to get the general solution does not exist. In other words, we could also obtain the general solution in the last case, but not by a straightforward process of separation of variables induced by the reduction procedure.

A further remark deserves the fact that the eigen-functions of a given operator (in our case the Hamiltonian $\hat{\mathcal{H}}$) define a Hilbert space. The above result works also in this case, so that we can define, for a quantum system whose eigenfunctions are given by a set of commuting Hermitian operators, a *Hilbert Space of General Conservation Laws* (see also [5]). The number of dimensions of such a space is given by the components of superposition (18) while the number of symmetries is given by the oscillatory components. Viceversa, the oscillatory components are *always* related to the number of symmetries in the corresponding Hilbert space. These results can be applied to minisuperspace quantum cosmology. The role of symmetries and conservation laws is prominent to interpret the information contained in the wave function of the universe which is solution of the WDW equation; in fact, the conserved momenta, related to some (or all) of the physical variables defining the minisuperspace, select oscillatory behaviors (*i.e.* strong peaks) in Ψ, which means "correlation" among the physical variables and then classical trajectories whose interpretation is that of "*observable universes*". In this sense, the so called Hartle criterion of quantum cosmology becomes a sufficient and necessary condition to select classical universes among all those which are possible. Working out this approach, we obtain the wave function of the universe Ψ depending on a set of physical parameter which are D, the initial bulk of matter, k, the spatial curvature constant, γ, the adiabatic index of perfect fluid matter, Λ, the parameter of the interaction potential.

The approach allows to recover several classes of interesting cosmological behaviors as De Sitter-like-singularity free solutions, power-law solutions, and pole-like solutions [3].

However, some points have to be considered in the interpretation of the approach. The Hartle criterion works in the context of an Everett-type interpretation of Quantum Cosmology [9, 17] which assumes the idea that the universe branches into a large number of copies of itself whenever a measurement is made. This point of view is the so called *Many Worlds* interpretation of Quantum Cosmology. Such an interpretation is an approach which gives a formulation of quantum mechanics designed to deal with correlations internal to individual, isolated systems. The Hartle criterion gives an operative interpretation of such correlations. In particular, if the wave function is *strongly peaked* in some region of configuration space, the correlations which characterize such a region can be, in principle, observed. On the other hand, if the wave function is *smooth* in some region, the correlations which characterize that region are precluded to the observations (that is, the cosmological parameters as H_0 or Ω_Λ cannot be neither calculated nor observed).

If the wave function is neither peaked nor smooth, no predictions are possible from observations. In conclusion, the analogy with standard quantum mechanics is straightforward. By considering the case in which the individual system consists of a large number of identical subsystems, one can derive, from the above interpretation, the usual probabilistic interpretation of Quantum Mechanics for the subsystems [11, 10]. If a conservation law (or more than one) is present for a given minisuperspace model, then strongly peaked (oscillatory) subsets of the wave function of the universe are found. Viceversa, oscillatory parts of the wave function can be always connected to conserved momenta and then to symmetries.

References

1. Basini G., Capozziello S. *Gen. Relativ. Grav.*, 2005, v. 37, 115.
2. Basini G., Capozziello S. *Europhys. Lett.*, 2003, v. 63, 166.
3. Basini G., Bongiorno F., Capozziello S. *Int. Journ. Mod. Phys.*, 2004, v. D13, 717.
4. Basini G., Bongiorno F., Capozziello S., Ricci M. *Int. Journ. Mod. Phys.*, 2004, v. D13, 359.
5. Basini G., Capozziello S., Longo G. *Phys. Lett.*, 2003, v. 311A, 465.
6. Basini G., Capozziello S. *Gen. Relativ. Grav.*, 2003, v. 35, 2217.
7. Smarandache F. A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability. 3rd edition, American Research Press, Rehoboth, 2003 (recent editions occured in 1999 and 2000).
8. Mao Linfan. Smarandache multi-space theory. Post-doctoral thesis. Hexis, Phoenix, 2005.
9. Everett H. *Rev. Mod. Phys.*, 1957, v. 29, 454.
10. Halliwell J. J. *Nucl. Phys.*, 1986, v. B266, 228.
11. Hartle J. B. In: *Gravitation in Astrophysics*, Cargese, 1986, eds. B. Carter and J. B. Hartle, Plenum Press, New York, 1986.
12. DeWitt B. S. *Phys. Rev.*, 1967, v. 160, 1113.
13. Misner C. W. *Phys. Rev.*, 1969, v. 186, 1319.
14. Hartle J. B. and Hawking S. W. *Phys. Rev.*, 1983, v. D28, 2960.
15. Vilenkin A. *Phys. Rev.*, 1986, v. D33, 3560.
16. Linde A. D. *Phys. Lett.*, 1982, v. B108, 389.
17. Finkelstein D. *Trans. N. Y. Acad. Sci.*, 1963, v. 25, 621.