# Momentum of the Pure Radiation Field

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The local momentum equation of the pure radiation field is considered in terms of an earlier elaborated and revised electromagnetic theory. In this equation the contribution from the volume force is found to vanish in rectangular geometry, and to become nonzero but negligible in cylindrical geometry. Consequently the radiated momentum is due to the Poynting vector only, as in conventional electrodynamics. It results in physically relevant properties of a photon model having an angular momentum (spin). The Poynting vector concept is further compared to the quantized momentum concept for a free particle, as represented by a spatial gradient operator acting on the wave function. However, this latter otherwise successful concept leads to difficulties in the physical interpretation of known and expected photon properties such as the spin, the negligible loss of transverse momentum across a bounding surface, and the Lorentz invariance.

## 1 Introduction

In the original and current presentation of Quantum Electrodynamics, the Poynting vector forms a basis for the quantized momentum of the pure radiation field [1, 2]. Thereby Maxwell's equations with a vanishing electric field divergence in the vacuum state are used to determine the electromagnetic field strengths and their potentials which, in their turn, are expressed by sets of quantized plane waves.

In the deduction of the Schrödinger equation, the quantized momentum for a free particle with mass has on the other hand been represented by an operator acting on the wave function and including a spatial gradient [1].

Since the individual photon can appear both as a wave and as a particle, the question may be raised whether its momentum should be represented by the Poynting vector concept, or by the spatial gradient operator concept. This question is discussed and illustrated in the present paper, in terms of a revised electromagnetic theory described in a recent review [3]. A summary of the basic equations of the theory is presented in Section 2, followed by two simple examples in Section 3 on a slab-shaped dense photon beam and on an axisymmetric model of the individual photon. A comparison between the two momentum concepts is finally made in Section 4.

## 2 Basic equations of the revised theory

The zero-point-energy of the vacuum state, its related electromagnetic vacuum fluctuations, the Casimir effect, and the electron-positron pair formation out of the vacuum support the hypothesis of a local electric charge density and an associated nonzero electric field divergence in such a state. On account of this, a Lorentz and gauge invariant theory has been elaborated, the details of which are given elsewhere [3–8]. The basic equations for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ become

$$\operatorname{curl}\mathbf{B}/\mu_0 = \varepsilon_0(\operatorname{div}\mathbf{E})\mathbf{C} + \varepsilon_0\,\partial\mathbf{E}/\partial t\,, \tag{1}$$

$$\operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t\,, \tag{2}$$

$$\operatorname{div}\mathbf{E} = \bar{\rho}/\varepsilon_0\,. \tag{3}$$

Here $\bar{\rho}$ is the local electric charge density in the vacuum, $\varepsilon_0$ and $\mu_0$ are the conventional dielectric constant and magnetic permeability of the vacuum, $c^2 = 1/\mu_0\varepsilon_0$, and $\mathbf{C}^2 = c^2$ results from the Lorentz invariance where $\mathbf{C}$ has the character of a velocity vector. Combination of equations (1) and (2) yields the extended wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\mathbf{E} + \left(c^2\nabla + \mathbf{C}\frac{\partial}{\partial t}\right)(\operatorname{div}\mathbf{E}) = 0 \tag{4}$$

for the electric field, and the associated relation

$$\left(\frac{\partial}{\partial t} + \mathbf{C}\cdot\nabla\right)(\operatorname{div}\mathbf{E}) = 0 \tag{5}$$

provided that $\operatorname{div}\mathbf{C} = 0$ which is an adopted restriction henceforth.

Using known vector identities, the basic equations (1), (2), and (3) result in the local momentum equation

$$\operatorname{div}{}^2\mathbf{S} = \mathbf{f} + \frac{\partial}{\partial t}\mathbf{g}\,, \tag{6}$$

where ${}^2\mathbf{S}$ is the electromagnetic stress tensor,

$$\mathbf{f} = \bar{\rho}\,\mathbf{E}' \qquad \mathbf{E}' = \mathbf{E} + \mathbf{C}\times\mathbf{B} \tag{7}$$

is the local volume force density, and

$$\mathbf{g} = \varepsilon_0\,\mathbf{E}\times\mathbf{B} = \frac{1}{c^2}\,\mathbf{S} \tag{8}$$

can be interpreted as a local electromagnetic momentum density of the radiation field, with $\mathbf{S}$ standing for the Poynting vector. Likewise a local energy equation

$$-\operatorname{div}\mathbf{S} = \bar{\rho}\,\mathbf{E}\cdot\mathbf{C} + \frac{1}{2}\,\varepsilon_0\,\frac{\partial}{\partial t}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \qquad (9)$$

is obtained. It is here to be observed that equations (6) and (9) are rearranged relations which do not provide more information than the original basic equations.

In the examples to be considered here, a velocity vector of the form

$$\mathbf{C} = c\,(0, \cos\alpha, \sin\alpha) \qquad (10)$$

is adopted, either in a rectangular frame $(x, y, z)$ or in a cylindrical frame $(r, \varphi, z)$. All field quantities are assumed to vary with $t$ and $z$ as $\exp\left[i(-\omega t + kz)\right]$ where $\omega$ and $k$ are the corresponding frequency and wave number of an elementary normal mode. Equation (5) then results in the dispersion relation

$$\omega = kv \qquad v = c(\sin\alpha)\,. \qquad (11)$$

In order not to get in conflict with observations, such as those due to the Michelson-Morley experiments, the analysis is restricted to the condition

$$0 < \cos\alpha \equiv \delta \ll 1\,. \qquad (12)$$

With a smallness parameter $\delta \leqslant 10^{-4}$, the difference between $v$ and $c$ would become less than a change in the eight decimal of $c$.

## 3 Normal modes in slab-shaped and axisymmetric geometries

The first example is given by a slab-shaped dense light beam. The beam propagates in the $z$-direction of a rectangular frame $(x, y, z)$, has a core region defined by $-a < x < a$, and two narrow boundary regions at $-b < x < -a$ and $a < x < b$. Within the core there is a homogeneous conventional electromagnetic wave field. This field is matched to the electromagnetic field in the inhomogeneous boundary regions as shown elsewhere [3, 8]. The analysis is here restricted to these regions within which the inhomogeneity in the $x$-direction requires the revised field equations to be used. In an analogous beam of circular cross-section, the source of angular momentum becomes localized to a corresponding inhomogeneous boundary region [3, 8].

The wave equation (4) now results in the relations

$$E_x = -(i/k\delta^2)\,\frac{\partial E_z}{\partial x}\,, \qquad (13)$$

$$E_y = -(\sin\alpha)\,E_z/\delta\,, \qquad (14)$$

where the field $E_z$ plays the rôle of a generating function for the components $E_x$ and $E_y$. From equation (2) the magnetic field components become

$$B_x = -E_y/c(\sin\alpha)\,, \qquad (15)$$

$$B_y = E_x/c(\sin\alpha) + \frac{i}{k\,c(\sin\alpha)}\,\frac{\partial E_z}{\partial x} = (\sin\alpha)E_x/c\,, \qquad (16)$$

$$B_z = -\frac{i}{k\,c(\sin\alpha)}\,\frac{\partial E_y}{\partial x} = -\delta E_x/c\,. \qquad (17)$$

Insertion of relations (13)–(17) into the expression (7) for the volume force then yields $\mathbf{E}' = 0$.

Further turning to the momentum density (8) of the radiation field, relations (13)–(17) give

$$g_x = 0\,, \qquad (18)$$

$$g_y = \delta\varepsilon_0\left[E_x^2 + E_y^2/(\sin\alpha)^2\right]/c\,, \qquad (19)$$

$$g_z = \varepsilon_0\left[E_x^2 + E_y^2/(\sin\alpha)^2\right]/c\,. \qquad (20)$$

Finally the power term in the energy equation (9) vanishes because relations (10), (13), and (14) combine to

$$\mathbf{E}\cdot\mathbf{C} = 0\,. \qquad (21)$$

This example thus demonstrates the following features:

- The volume force density $\mathbf{f}$ vanishes in rectangular geometry.
- The momentum density $\mathbf{g}$ of the radiation field has a primary component $g_z$ in the direction of propagation.
- There is a secondary component $g_y$ of the order $\delta$, directed along the boundary and being perpendicular to the direction of propagation. This component corresponds to that which generates angular momentum (spin) in cylindrical geometry.
- There is a vanishing component $g_x$ and no momentum is flowing across the boundary of the beam.
- The local power term in the energy equation vanishes.

The second example concerns an axisymmetric model of the individual photon. A wave or a wave packet of preserved and limited geometrical shape and undamped motion in a defined direction has then to be taken as a starting point. This leads to cylindrical geometry with propagation along $z$ in a frame $(r, \varphi, z)$. From earlier deductions based on equations (1)–(5), the electric and magnetic field components of an elementary normal mode then become [3–6]

$$E_r = -i\,g_0\,R_5/\theta\,, \qquad (22)$$

$$E_\varphi = g_0\,\delta(\sin\alpha)R_3\,, \qquad (23)$$

$$E_z = g_0\,\delta^2 R_4 \qquad (24)$$

and

$$B_r = -E_\varphi/c(\sin\alpha) = -g_0\,\delta R_3/c\,, \qquad (25)$$

$$B_\varphi = E_r(\sin\alpha)/c = -i\,g_0\,(\sin\alpha)R_5/\theta c\,, \qquad (26)$$

$$B_z = -i\,g_0\,\delta R_8/\theta c\,. \qquad (27)$$

Here we have introduced $g_0 = G_0/\delta^2$ where $G_0$ is the characteristic amplitude of a normalized generating function $G$, $\theta = kr_0$ with $r_0$ as a characteristic radial length, and

$$R_3 = \rho^2 DG \qquad R_4 = 1 - R_3 \tag{28}$$

$$R_5 = \frac{\partial}{\partial\rho} R_4 \qquad R_8 = \left(\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right) R_3 \tag{29}$$

with $\rho = r/r_0$ and the operator $D$ given by

$$D = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \theta^2\delta^2 \,. \tag{30}$$

For the electric field $\mathbf{E}'$ the components now reduce to

$$E'_r = -i g_0 \delta^2 (R_5 + R_8)\,, /\theta \tag{31}$$

$$E'_\varphi = 0\,, \tag{32}$$

$$E'_z = g_0 \delta^2 (R_3 + R_4) \tag{33}$$

and the momentum components of the radiation field are given by

$$c g_r/\varepsilon_0 g_0^2 = i\delta^2 (\sin\alpha)(R_4 R_5 - R_3 R_8)/\theta\,, \tag{34}$$

$$c g_\varphi/\varepsilon_0 g_0^2 = \delta R_5 R_8/\theta^2 - \delta^3 R_3 R_4\,, \tag{35}$$

$$c g_z/\varepsilon_0 g_0^2 = -(\sin\alpha) R_5^2/\theta^2 + \delta^2 (\sin\alpha) R_3^2\,. \tag{36}$$

Finally the power term in the energy equation (9) becomes

$$\bar\rho\,\mathbf{E}\cdot\mathbf{C} = \delta^2 \bar\rho\, c g_0 (\sin\alpha)(R_3 + R_4) \tag{37}$$

thus being of second order in the parameter $\delta$.

To the first order in $\delta$ the axisymmetric geometry then has features being analogous to those of the slab-shaped geometry:

- There is a negligible contribution from the volume force density $\mathbf{f}$, as well as from the radial component $g_r$ of the radiation field.
- A secondary component $g_\varphi$ of order $\delta$ gives rise to a spin of the photon model [3].
- The power term in the energy equation is negligible.

A corresponding analysis of a non-axisymmetric photon model with periodic $\varphi$-dependence and screw-shaped geometry leads to similar results [7].

The total(net) electric charge and magnetic moment of the present photon models have finally been shown to vanish through spatial integration [5–7].

## 4 Comparison between the momentum concepts

In the spatial gradient concept the momentum is represented by the operator

$$\mathbf{p} = -i\hbar\nabla\,. \tag{38}$$

For the normal modes being considered here, the corresponding axial component reduces to

$$p_z = \hbar k = h/\lambda = h\nu/c \tag{39}$$

which in conventional theory becomes related to a photon of energy $h\nu$, moving along $z$ at the velocity $c$ of light.

A comparison between the concepts of equations (8) and (38) is now made in respect to the remaining components being perpendicular to the direction of propagation, as well as in respect to the related question about Lorentz invariance.

### 4.1 The transverse component directed across a confining boundary

As compared to the axial component $g_z$, the momentum density $\mathbf{g}$ has a vanishing component $g_x$ in slab-shaped geometry, and a nonzero but negligible component $g_r$ in axisymmetric geometry. The corresponding relations between the momentum $p_z$ and the components $p_x$ and $p_r$ are in a first approximation represented by

$$|p_x/p_z| \cong \lambda/2\pi L_x\,, \qquad |p_r/p_z| \cong \lambda/2\pi L_r \tag{40}$$

with $L_x$ and $L_r$ as corresponding characteristic lengths. Then the transverse components $p_x$ and $p_r$ cannot generally be neglected. This becomes questionable from the physical point of view when considering individual photons and light beams which have no transverse losses of momentum.

### 4.2 The transverse component directed along a confining boundary

With vanishing derivatives $\partial/\partial y$ or $\partial/\partial\varphi$, along a boundary in rectangular geometry or around the axis in cylindrical geometry, there are components $g_y$ and $g_\varphi$ being related to a nonzero spin. This behaviour differs from that of the momentum $\mathbf{p}$ for which the components $p_y$ and $p_\varphi$ vanish, as well as the spin. Such a behaviour appears to lack physical explanation.

When there are nonvanishing derivatives $\partial/\partial y$ and $\partial/\partial\varphi$, the concepts of $\mathbf{g}$ and $\mathbf{p}$ both result in transverse components along a boundary, but being of different forms.

### 4.3 The Lorentz invariance

In the present revised Lorentz invariant theory on the photon model, there is a component of the momentum $\mathbf{g}$ around the axis. This provides a spin, at the expense of the axial velocity of propagation. The latter then has to become slightly less than $c$, as required by the dispersion relation (11).

With the definition (38) of the momentum $\mathbf{p}$, there is a different situation. Thus equation (39) is in conventional theory consistent with an individual photon that moves at the full velocity $c$ along the axial direction. But for the same photon also to possess a nonzero spin, it should have an additional transverse momentum $p_\varphi$, with an associated velocity $v_\varphi$ which circulates around the $z$-axis. For a radiation field within the volume of the photon to be considered as a self-consistent entity, the total local velocity then becomes equal to $(c^2+v_\varphi^2)^{1/2} > c$. This would represent a superluminal field configuration not being Lorentz invariant.

## 5 Conclusions

As expressed in terms of the present revised electromagnetic theory, the momentum concept of the pure radiation field appears to be physically relevant. The corresponding volume force density thus vanishes in rectangular geometry and is nonzero but negligible in cylindrical geometry. The momentum density is represented by the Poynting vector, as in conventional theory. Thereby its transverse components become consistent with the spin of the photon, and with a negligible loss of transverse momentum across a bounding surface.

The spatial gradient operator concept for the quantized momentum of a free particle with mass has earlier been used with success in the Schrödinger equation. However, when applying this concept to the free radiation field of the individual photon or of dense light beams, the obtained results differ from those based on the Poynting vector, and are in some cases difficult to interpret from the physical point of view. This discrepancy requires further investigation.

In this connection it should finally be mentioned that the present axisymmetric photon model [3, 6] is radially polarized. The core of a dense light beam being treated earlier [8] consists on the other hand of a linearly polarized conventional electromagnetic wave, with a boundary region having a radial gradient and leading to a spin of the beam considered as an entity.

The theory of this latter model can as well be applied to the limit of an individual photon with a linearly polarized core, a boundary region of finite radial extension, and a nonzero spin. It should thereby be kept in mind that such a model concerns the internal structure of a single photon, and therefore does not deal with the entangled quantum states of two interacting photons.

## References


1. Schiff L. I. Quantum Mechanics. McGraw-Hill Book Comp. Inc., New York, 1949, Ch. XIV, Ch. II.
2. Heitler W. The quantum theory of radiation. Third edition. Clarendon Press, Oxford, 1954, Ch. II.
3. Lehnert B. Photon physics of revised electromagnetics. *Progress in Physics*, 2006, v. 2, 78–85.
4. Lehnert B., Roy S. Extended electromagnetic theory. World Scientific Publishers, Singapore, 1998.
5. Lehnert B. Optical effects of an extended electromagnetic theory. In: *Advances in Chemical Physics*, v. 119, Edited by M. W. Evans, I. Prigogine, and S. A. Rice, John Wiley and Sons, Inc., New York, 2001, 1–77.
6. Lehnert B. Photon wave and particle concepts of an extended electromagnetic theory. *Physica Scripta*, 2002, v. 66, 105–113.
7. Lehnert B. Screw-shaped light in extended electromagnetics. *Physica Scripta*, 2005, v. 72, 359–365.
8. Lehnert B. Boundary conditions and spin of a dense light beam. *Physica Scripta*, 2006, v. 74, 139–144.