The Algebraic Rainich Conditions

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In the literature, the algebraic Rainich conditions are obtained using special methods such as spinors, duality rotations, an eigenvalue problem for certain 4 × 4 matrices or artificial tensors of 4th order. We give here an elementary procedure for deducing an identity satisfied by a determined class of second order tensors in arbitrary ℜ^n, from which the Rainich expressions are immediately obtained.

1 Introduction

Rainich [1–5] proposed a unified field theory for the geometrization of the electromagnetic field, whose basic relations can be obtained from the Einstein-Maxwell field equations:

\[ R_{ij} - \frac{1}{2} R g_{ij} = -8 \pi \left( F_{ib} F^b_j - \frac{1}{4} F_{ab} F^{ab} g_{ij} \right) , \]  (1)

where \( R_{ac} = R_{ca} , R = R^b_b \) and \( F_{ac} = -F_{ca} \) are the Ricci tensor, scalar curvature and Faraday tensor [6], respectively.

If in (1) we contract \( i \) with \( j \) we find that:

\[ R = 0 \]  (2)

then (1) adopts the form:

\[ R_{ij} = 2 \pi F_{ab} F^{ab} g_{ij} - 8 \pi F_{ib} F^b_j , \]  (3)

used by several authors [1, 2, 5, 7, 8] to obtain the identity:

\[ R_{ic} R^c_j = \frac{1}{4} \left( R_{ab} F^{ab} \right) g_{ij} . \]  (4)

If \( F_{ar} \) is known, then (3) is an equation for \( g_{ij} \) and our situation belongs to general relativity. The Rainich theory presents the inverse process: To search for a solution of (2) and (4) (plus certain differential restrictions), and after with (3) to construct the corresponding electromagnetic field; from this point of view \( F_{ar} \) is a consequence of the spacetime geometry.

In the next Section we give an elementary proof of (4), without resorting to duality rotations [2], spinors [7], eigenvalue problems [8] or fourth order tensors [9, 10].

2 The algebraic Rainich conditions

The structure of (3) invites us to consider tensors with the form:

\[ C_{ij} = A g_{ij} + B_{ik} F^k_j , \]  (5)

where \( A \) is a scalar and \( B_{ac} , F_{ij} \) are arbitrary antisymmetric tensors. Then from (5) it is easy to deduce the expression:

\[ C_{ia} C^{a}_j - \frac{C}{2} C_{ij} - \frac{1}{4} \left( C_{ab} C^{ba} - \frac{C^2}{2} \right) g_{ij} = D_{ij} \]  (6)

with \( C = C^\tau_\tau \) and

\[ D_{ij} = B_{ik} F^{ak} B_{am} F^m_j - \frac{1}{2} \left( B^{nm} F_{nm} \right) B_{ab} F^b_j + \frac{1}{8} \left[ \left( B^{nm} F_{nm} \right)^2 - 2 B_{bk} F^k_a B_a^m \right] g_{ij} . \]  (7)

But in four dimensions we have the following identities between antisymmetric tensors and their duals [11–13]:

\[ B_{ci} F^{ic} - * B_{ci} * F^m_c = \frac{1}{2} \left( B_{cd} F^{cd} \right) g^{im} , \]  (8)

\[ B^*_k * B^* r = \frac{1}{4} \left( B_{ab} * B^{ab} \right) g^{kr} . \]

With (7) and (8) it is simple to prove that \( D_{ij} = 0 \). Therefore (6) implies the identity:

\[ C_{ia} C^{a}_j - \frac{C}{2} C_{ij} = \frac{1}{4} \left( C_{ab} C^{ba} - \frac{C^2}{2} \right) g_{ij} . \]  (9)

If now we consider the particular case:

\[ A = 2 \pi F_{ab} F^{ab} , \quad B_{ij} = -8 \pi F_{ij} , \]  (10)

then (5) reproduces (3) and \( C = R = 0 \), and thus (9) leads to (4), q.e.d.

Our procedure shows that the algebraic Rainich conditions can be deduced without special techniques.

References


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