Remarks on Conformal Mass and Quantum Mass

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One shows how in certain model situations conformal general relativity corresponds to a Bohmian-Dirac-Weyl theory with conformal mass and Bohmian quantum mass identified.

The article [12] was designed to show relations between conformal general relativity (CGR) and Dirac-Weyl (DW) theory with identification of conformal mass $\hat{m}$ and quantum mass $\mathfrak{M}$ following [7, 9, 11, 25] and precision was added via [21]. However the exposition became immersed in technicalities and details and we simplify matters here. Explicitly we enhance the treatment of [7] by relating $\mathfrak{M}$ to an improved formula for the quantum potential based on [21] and we provide a specific Bohmian-Dirac-Weyl theory wherein the identification of CGR and DW is realized. Much has been written about these matters and we mention here only [1–7, 9–20, 23–28] and references therein. One has an Einstein form for GR of the form

$$S_{GR} = \int d^4x\,\sqrt{-g}(R - \alpha|\nabla\psi|^2 + 16\pi L_M) \tag{1.1}$$

(cf. [7, 22]) whose conformal form (conformal GR) is an integrable Weyl geometry based on

$$\begin{aligned}\hat{S}_{GR} &= \int d^4x\,\sqrt{-\hat{g}}\;e^{-\psi}\;\times\\ &\times\left[\hat{R} - \left(\alpha - \frac{3}{2}\right)|\hat{\nabla}\psi|^2 + 16\pi e^{-\psi}L_M\right] = \\ &= \int d^4x\,\sqrt{-\hat{g}}\left[\hat{\phi}\hat{R} - \left(\alpha-\frac{3}{2}\right)\frac{|\hat{\nabla}\hat{\phi}|^2}{\hat{\phi}} + 16\pi\hat{\phi}^2 L_M\right]\end{aligned} \tag{1.2}$$

where $\Omega^2 = \exp(-\psi) = \phi$ with $\hat{g}_{ab} = \Omega^2 g_{ab}$ and $\hat{\phi} = \exp(\psi) = \phi^{-1}$ (note $(\hat{\nabla}\psi)^2 = (\hat{\nabla}\hat{\phi})^2/(\hat{\phi})^2$). One sees also that (1.2) is the same as the Brans-Dicke (BD) action when $L_M = 0$, namely (using $\hat{g}$ as the basic metric)

$$S_{BD} = \int d^4x\,\sqrt{-\hat{g}}\left[\hat{\phi}\hat{R} - \frac{\omega}{\hat{\phi}}|\hat{\nabla}\hat{\phi}|^2 + 16\pi L_M\right]; \tag{1.3}$$

which corresponds to (1.2) provided $\omega = \alpha - \frac{3}{2}$ and $L_M = 0$. For (1.2) we have a Weyl gauge vector $w_a \sim \partial_a\psi = \partial_a\hat{\phi}/\hat{\phi}$ and a conformal mass $\hat{m} = \hat{\phi}^{-1/2}m$ with $\Omega^2 = \hat{\phi}^{-1}$ as the conformal factor above. Now in (1.2) we identify $\hat{m}$ with the quantum mass $\mathfrak{M}$ of [25] where for certain model situations $\mathfrak{M} \sim \beta$ is a Dirac field in a Bohmian-Dirac-Weyl theory as in (1.8) below with quantum potential Q determined via $\mathfrak{M}^2 = m^2\exp(Q)$ (cf. [10, 11, 21, 25] and note that $m^2 \propto T$ where $8\pi T^{ab} = (1/\sqrt{-g})(\delta\sqrt{-g}\,\mathfrak{L}_M/\delta g_{ab})$). Then $\hat{\phi}^{-1} = \hat{m}^2/m^2 = \mathfrak{M}^2/m^2 \sim \Omega^2$ for Ω^2 the standard conformal factor of [25]. Further one can write (**1A**) $\sqrt{-\hat{g}}\,\hat{\phi}\hat{R} = \hat{\phi}^{-1}\sqrt{-\hat{g}}\,\hat{\phi}^2\hat{R} = \hat{\phi}^{-1}\sqrt{-g}\,\hat{R} = (\beta^2/m^2)\sqrt{-g}\,\hat{R}$. Recall here from [11] that for $g_{ab} = \hat{\phi}\hat{g}_{ab}$ one has $\sqrt{-g} = \hat{\phi}^2\sqrt{-\hat{g}}$ and for the Weyl-Dirac geometry we give a brief survey following [11, 17]:

1. Weyl gauge transformations: $g_{ab} \to \tilde{g}_{ab} = e^{2\lambda}g_{ab}$; $g^{ab} \to \tilde{g}^{ab} = e^{-2\lambda}g^{ab}$ — weight e.g. $\Pi(g^{ab}) = -2$. β is a Dirac field of weight -1. Note $\Pi(\sqrt{-g}) = 4$;
2. Γ^c_{ab} is Riemannian connection; Weyl connection is $\hat{\Gamma}^c_{ab}$ and $\hat{\Gamma}^c_{ab} = \Gamma^c_{ab} = g_{ab}w^c - \delta^c_b w_a - \delta^c_a w_b$;
3. $\nabla_a B_b = \partial_a B_b - B_c\Gamma^c_{ab}$; $\nabla_a B^b = \partial_a B^b + B^c\Gamma^b_{ca}$;
4. $\hat{\nabla}_a B_b = \partial_a B_b - B_c\hat{\Gamma}^c_{ab}$; $\hat{\nabla}_a B^b = \partial_a B^b + B^c\hat{\Gamma}^b_{ca}$;
5. $\hat{\nabla}_\lambda g^{ab} = -2g^{ab}w_\lambda$; $\hat{\nabla}_\lambda g_{ab} = 2g_{ab}w_\lambda$ and for $\Omega^2 = \exp(-\psi)$ the requirement $\nabla_c g_{ab} = 0$ is transformed into $\hat{\nabla}_c\hat{g}_{ab} = \partial_c\psi\hat{g}_{ab}$ showing that $w_c = -\partial_c\psi$ (cf. [7]) leading to $w_\mu = \hat{\phi}_\mu/\hat{\phi}$ and hence via $\beta = m\hat{\phi}^{-1/2}$ one has $w_c = 2\beta_c/\beta$ with $\hat{\phi}_c/\hat{\phi} = -2\beta_c/\beta$ and $w^a = -2\beta^a/\beta$.

Consequently, via $\beta^2\hat{R} = \beta^2 R - 6\beta^2\nabla_\lambda w^\lambda + 6\beta^2 w^\lambda w_\lambda$ (cf. [11, 12, 16, 17]), one observes that $-\beta^2\nabla_\lambda w^\lambda = -\nabla_\lambda(\beta^2 w^\lambda) + 2\beta\partial_\lambda\beta w^\lambda$, and the divergence term will vanish upon integration, so the first integral in (1.2) becomes

$$I_1 = \int d^4x\,\sqrt{-g}\left[\frac{\beta^2}{m^2}R + 12\beta\partial_\lambda\beta w^\lambda + 6\beta^2 w^\lambda w_\lambda\right]. \tag{1.4}$$

Setting now $\alpha - \frac{3}{2} = \gamma$ the second integral in (1.2) is

$$\begin{aligned}I_2 &= -\gamma\int d^4x\,\sqrt{-\hat{g}}\;\hat{\phi}\,\frac{|\hat{\nabla}\hat{\phi}|^2}{|\hat{\phi}|^2} = \\ &= -4\gamma\int d^4x\,\sqrt{-\hat{g}}\;\hat{\phi}^{-1}\hat{\phi}^2\,\frac{|\hat{\nabla}\beta|^2}{\beta^2} = \\ &= -\frac{4\gamma}{m^2}\int d^4x\,\sqrt{-g}|\hat{\nabla}\beta|^2,\end{aligned} \tag{1.5}$$

while the third integral in the formula (1.2) becomes (**1B**) $16\pi\int\sqrt{-g}\,d^4x L_M$. Combining now (1.4), (1.5), and (**1B**) gives then

$$\begin{aligned}\hat{S}_{GR} = \frac{1}{m^2}\int d^4x\,\sqrt{-g}\,[\beta^2 R + 6\beta^2 w^\alpha w_\alpha + \\ + 12\beta\partial_\alpha\beta w^\alpha - 4\gamma|\hat{\nabla}\beta|^2 + 16\pi m^2 L_M].\end{aligned} \tag{1.6}$$

We will think of $\hat{\nabla}\beta$ in the form (**1C**) $\hat{\nabla}_\mu\beta = \partial_\mu\beta - w_\mu\beta = -\partial_\mu\beta$. Putting then $|\hat{\nabla}\beta|^2 = |\partial\beta|^2$ (1.6) becomes (recall $\gamma = \alpha - \frac{3}{2}$)

$$\hat{S}_{GR} = \frac{1}{m^2}\int d^4x\,\sqrt{-g}\,\times \times \left[\beta^2 R + (3-4\alpha)|\partial\beta|^2 + 16\pi m^2 L_M\right]. \tag{1.7}$$

One then checks this against some Weyl-Dirac actions. Thus, neglecting terms $W^{ab}W_{ab}$ we find integrands involving $dx^4\sqrt{-g}$ times

$$-\beta^2 R + 3(3\sigma+2)|\partial\beta|^2 + 2\Lambda\beta^4 + \mathfrak{L}_M \tag{1.8}$$

(see e.g. [11,12,17,25]); the term $2\Lambda\beta^4$ of weight -4 is added gratuitously (recall $\Pi(\sqrt{-g}) = 4$). Consequently, omitting the Λ term, (1.8) corresponds to (1.7) times m^2 for $\mathfrak{L}_M \sim \sim 16\pi L_M$ and (**1D**) $9\sigma + 4\alpha + 3 = 0$. Hence one can identify conformal GR (without Λ) with a Bohmian-Weyl-Dirac theory where conformal mass $\hat{m}$ corresponds to quantum mass $\mathfrak{M}$.

REMARK 1.1. The origin of a β^4 term in (1.8) from $\hat{S}_{GR}$ in (1.2) with a term $2\sqrt{-\hat{g}}\hat{\Lambda}$ in the integrand would seem to involve writing (**1E**) $2\sqrt{-\hat{g}}\,\hat{\Lambda} = 2\sqrt{-\hat{g}}\,\hat{\phi}^2\Omega^4\hat{\Lambda} = = 2\sqrt{-g}\,\beta^4\hat{\Lambda}/m^4$ so that Λ in (1.8) corresponds to $\hat{\Lambda}$. Normally one expects $\Lambda\sqrt{-g} \to \sqrt{-\hat{g}}\,\hat{\phi}^2\Lambda$ (cf. [2]) or perhaps $\Lambda \to \hat{\phi}^2\Lambda = \Omega^{-4}\Lambda = \hat{\Lambda}$. In any case the role and nature of a cosmological constant seems to still be undecided. ■

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