Third Quantization in Bergmann-Wagoner Theory

César Mora and Rubén Sánchez

Centro de Investigación en Ciencia Aplicada y Tecnología Avanzada,
Instituto Politécnico Nacional,
Av. Legaria 694, Col. Irrigación, CP 11500, México D. F., Mexico

E-mail: cmoral@ipn.mx, rsanchezs@ipn.mx

We present the third quantization of Bergmann-Wagoner scalar-tensor and Brans Dicke solvable toy models. In the first one we used an exponential cosmological term, for the second one we considered vanishing cosmological constant. In both cases, it is found that the number of the universes produced from nothing is very large.

1 Introduction

The Wheeler-DeWitt (WDW) equation is a result of quantization of a geometry and matter (second quantization of gravity), in this paper we consider the third quantization of a solvable inflationary universe model, i.e., by analogy with the quantum field theory, it can be done the second quantization of the universe wavefunction ψ expanding it on the creation and annihilation operators (third quantization) [1]. Because in the recent years there has been a great interest in the study of scalar-tensor theories of gravitation, owing that of the unified theories [2, 3], we choose to work with the most general scalar-tensor theory examined by Bergmann and Wagoner [4, 5], in this theory the Brans-Dicke parameter ω and cosmological function λ depend upon the scalar gravitational field ϕ. The Brans-Dicke theory can be obtained setting $\omega = \text{const}$ and $\lambda = 0$.

The WDW equation is obtained by means of canonical quantization of Hamiltonian H according to the standard canonical rule, this leads to a difficulty known as the problem of time [6]. Also, this equation has problems in its probabilistic interpretation. In the usual formulation of quantum mechanics a conserved positive-definite probability density is required for a consistent interpretation of the physical properties of a given system, and the universe in the quantum cosmology perspective, do not satisfied this requirement, because the WDW equation is a hyperbolic second order differential equation, there is no conserved positive-definite probability density as in the case of the Klein-Gordon equation, an alternative to this, is to regard the wavefunction as a quantum field in minisuperspace rather than a state amplitude [7].

The paper is organized as follows. In Section 2 we obtain the WDW equation. In Section 3 we show third quantization of the universe wavefunction using two complete set of modes for the most easy choice of factor ordering. Finally, Section 4 consists of conclusions.

2 Canonical formalism

Our starting point is the action of Bergmann-Wagoner scalar tensor theory

$$S = \frac{1}{l_p^2}\int_M \sqrt{-g}\left[\phi R^{(4)} - \frac{\omega(\phi)}{\phi}\, g^{\mu\nu}\,\phi_{,\mu}\phi_{,\nu} + 2\,\phi\,\lambda(\phi)\right] d^4x + \\ + \frac{2}{l_p^2}\int_{\partial M} \sqrt{h}\,\phi\, h_{ij} K^{ij}\, d^3x\,, \tag{1}$$

where $g = \det(g_{\mu,\nu})$, $\phi(t)$ is the conventional real scalar gravitational field, while l_p is the Planck length and $\lambda(\phi)$ is the cosmological term. The quantity $R^{(4)}$ is the scalar curvature of the Friedmann-Robertson-Walker theory, which is given, according to the theory, by

$$R^{(4)} = -\frac{6k}{a^2} - 6\frac{\dot a^2}{N^2 a^2} - 6\frac{\ddot a}{N^2 a} + 6\frac{\dot a \dot N}{N^3 a}\,. \tag{2}$$

The second integral in (1) is a surface term involving the induced metric h_{ij} and second fundamental form K^{ij} on the boundary, needed to cancel the second derivatives in $R^{(4)}$ when the action is varied with the metric and scalar field, but not their normal derivatives, fixed on the boundary. Substituting (2) in (1) and integrating with respect to space coordinates, we have

$$S = \frac{1}{2}\int \left[-Nka\phi + \frac{a\phi}{N}\,\dot a^2 + \frac{a^2}{N}\,\dot a\dot\phi - \right. \\ \left. - \frac{N\omega(\phi)}{6\phi}\, a^3\dot\phi^2 + \frac{N}{3}\, a^3\phi\lambda(\phi)\right] dt\,, \tag{3}$$

where dot denotes time derivative with respect to the time t, now introducing a new time $d\tau = \phi^{\frac{1}{2}}\, dt$ and the following independent variables

$$\alpha = a^2\phi\cosh\int\left(\frac{2\omega(\phi)+3}{3}\right)^{\frac{1}{2}}\frac{d\phi}{\phi}\,, \tag{4}$$

$$\beta = a^2\phi\sinh\int\left(\frac{2\omega(\phi)+3}{3}\right)^{\frac{1}{2}}\frac{d\phi}{\phi}\,, \tag{5}$$

$$\lambda(\phi) = 3\phi\left[\Lambda_1\cosh\int\left(\frac{2\omega(\phi)+3}{3}\right)^{\frac{1}{2}}\frac{d\phi}{\phi} + \right. \\ \left. + \Lambda_2\sinh\int\left(\frac{2\omega(\phi)+3}{3}\right)^{\frac{1}{2}}\frac{d\phi}{\phi}\right], \tag{6}$$

where Λ_1 and Λ_2 are constants, with gauge $N = 1$, then action (3) transforms into a symmetric form

$$S = \frac{1}{2}\int \left[\frac{1}{4}\left(\alpha'^2 - \beta'^2\right) + \Lambda_1\alpha + \Lambda_2\beta - k\right] d\tau, \qquad (7)$$

here prime denotes time derivative with respect to τ. The Hamiltonian of the system is

$$H = 2\pi_\alpha^2 - 2\pi_\beta^2 + \frac{1}{2}(k - \Lambda_1\,\alpha - \Lambda_2\beta). \qquad (8)$$

After canonical quantization of H, the WDW equation is

$$\Big[\partial_\alpha^2 + A\alpha^{-1}\partial_\alpha - \partial_\beta^2 - B\beta^{-1}\partial_\beta + + \frac{1}{4}(\Lambda_1\,\alpha + \Lambda_2\beta - k)\Big]\psi(\alpha,\beta) = 0, \qquad (9)$$

where A and B are ambiguity ordering parameters. The general universe wavefunction for this model can be given in terms of Airy functions.

3 Third quantization

The procedure of the universe wavefunction ψ quantization is called third quantization, in this theory we consider ψ as an operator acting on the state vectors of a system of universes and can be decomposed as

$$\hat\psi(\alpha,\beta) = \hat C_i\,\psi_i^+(\alpha,\beta) + \hat C_i^\dagger\,\psi_i^-(\alpha,\beta)\,, \qquad (10)$$

where $\psi_i^\pm(\alpha,\beta)$ form complete orthonormal sets of solutions to WDW equation. This is in analogy with the quantum field theory, where $\hat C_i$ and $\hat C_i^\dagger$ are creation and annihilation operators. Thus, we expect that the vacuum state in a third quantized theory is unstable and creation of universes from the initial vacuum state takes place. In this view, the variable α plays the role of time, and variable β the role of space. $\psi(\alpha,\beta)$ is interpreted as a quantum field in the minisuperspace.

We assume that the creation and annihilation operators of universes obey the standard commutation relations

$$\left[\,C(s), C^\dagger(s')\right] = \delta(s - s'), \qquad (11)$$

$$\left[\,C(s), C(s')\right] = \left[\,C^\dagger(s), C^\dagger(s')\right] = 0\,. \qquad (12)$$

The vacuum state $|0\rangle$ is defined by

$$C(s)|0\rangle \quad \text{for} \quad \forall\, C\,, \qquad (13)$$

and the Fock space is spanned by $C^\dagger(s_1)C^\dagger(s_2)...|0\rangle$. The field $\psi(\alpha,\beta)$ can be expanded in normal modes ψ_s as

$$\psi(\alpha,\beta) = \int_{-\infty}^{+\infty}\left[\,C(s)\,\psi_s(\alpha,\beta) + C^\dagger(s)\,\psi_s^*(\alpha,\beta)\right]ds\,, \qquad (14)$$

here, the wave number s is the momentum in Planck units and is very small.

3.1 General model

Let us consider the quantum model (9) for the most easy factor ordering $A = B = 0$, with $\Lambda_2 = 0$ and closed universe $k = 1$. Then, the WDW equation becomes

$$\left[\partial_\alpha^2 - \partial_\beta^2 + \frac{1}{4}(\Lambda_1\alpha - 1)\right]\psi(\alpha,\beta) = 0, \qquad (15)$$

the third-quantized action to yield this equation is

$$S_{3Q} = \frac{1}{2}\int\left[(\partial_\alpha\psi)^2 - (\partial_\beta\psi)^2 - \frac{1}{4}(\Lambda_1\alpha - 1)\psi^2\right]d\alpha\, d\beta\,, \qquad (16)$$

this action can be canonically quantized and we impose the equal time commutation relations

$$\left[\,i\,\partial_\alpha\psi(\alpha,\beta), \psi(\alpha,\beta')\right] = \delta(\beta - \beta')\,, \qquad (17)$$

$$\left[\,i\,\partial_\alpha\psi(\alpha,\beta), i\,\partial_\alpha\psi(\alpha,\beta')\right] = 0\,, \qquad (18)$$

$$\left[\,\psi(\alpha,\beta), \psi(\alpha,\beta')\right] = 0\,. \qquad (19)$$

A suitable complete set of normalized positive frequency solutions to equation (15) are:

$$\psi_s^{out}(\alpha,\beta) = \frac{e^{is\beta}}{(16\Lambda_1)^{\frac{1}{16}}}\Big\{\mathrm{Ai}\left[(2\Lambda_1)^{-\frac{2}{3}}(1 - 4s^2 - \Lambda_1\alpha)\right] + + i\,\mathrm{Bi}\left[(2\Lambda_1)^{-\frac{2}{3}}(1 - 4s^2 - \Lambda_1\alpha)\right]\Big\}, \qquad (20)$$

and

$$\psi_s^{in}(\alpha,\beta) = \frac{\sqrt{2}\,e^{is\beta}}{(16\Lambda_1)^{\frac{1}{16}}} \times \times\Big\{e^{\frac{(1-4s^2)^{\frac{3}{2}}}{3\Lambda_1}}\mathrm{Ai}\left[(2\Lambda_1)^{-\frac{2}{3}}(1 - 4s^2 - \Lambda_1\alpha)\right] + + \frac{i}{2}\,e^{-\frac{(1-4s^2)^{\frac{3}{2}}}{3\Lambda_1}}\mathrm{Bi}\left[(2\Lambda_1)^{-\frac{2}{3}}(1 - 4s^2 - \Lambda_1\alpha)\right]\Big\}, \qquad (21)$$

$\psi_s^{out}(\alpha,\beta)$ and $\psi_s^{in}(\alpha,\beta)$ can be seen as a positive frequency out going and in going modes, respectively, and these solutions are orthonormal with respect to the Klein-Gordon scalar product

$$\langle\psi_s, \psi_{s'}\rangle = i\int\psi_s\,\overset{\leftrightarrow}{\partial}_\beta\,\psi_{s'}^*\,d\beta = \delta(s - s')\,. \qquad (22)$$

The expansion of $\psi(\alpha,\beta)$ in terms of creation and annihilation operators for the in-mode and out-mode is

$$\psi(\alpha,\beta) = \int\left[C_{in}(s)\,\psi_s^{in}(\alpha,\beta) + + C_{in}^\dagger(s)\,\psi_r^{in*}(\alpha,\beta)\right]ds\,, \qquad (23)$$

and

$$\psi(\alpha,\beta) = \int\left[C_{out}(s)\,\psi_s^{out}(\alpha,\beta) + + C_{out}^\dagger(s)\,\psi_r^{out*}(\alpha,\beta)\right]ds\,. \qquad (24)$$

As both sets (20) and (21) are complete, they are related to each other by the Bogoliubov transformation defined by

$$\psi_s^{out}(\alpha,\beta) = \int \left[\, C_1(s,r)\, \psi_r^{in}(\alpha,\beta) \; + \right. \\ \left. +\; C_2(s,r)\, \psi_r^{in*}(\alpha,\beta) \right] dr\,, \qquad (25)$$

and

$$\psi_s^{in}(\alpha,\beta) = \int \left[\, C_1(s,r)\, \psi_r^{out}(\alpha,\beta) \; + \right. \\ \left. +\; C_2(s,r)\, \psi_r^{out*}(\alpha,\beta) \right] dr\,. \qquad (26)$$

Then, we obtain that the Bogoliubov coefficients $C_1(s,r) = \delta(s-r)C_1(s)$ and $C_2(s,r) = \delta(s+r)C_2(s)$ are

$$C_1(s,r) = \delta(s-r)\frac{1}{\sqrt{2}} \times \\ \times \left(\frac{1}{2}\, e^{-\frac{(1-4s^2)^{\frac{3}{2}}}{3\Lambda_1}} + e^{\frac{(1-4s^2)^{\frac{3}{2}}}{3\Lambda_1}} \right), \qquad (27)$$

and

$$C_2(s,r) = \delta(s+r)\frac{1}{\sqrt{2}} \times \\ \times \left(\frac{1}{2}\, e^{-\frac{(1-4s^2)^{\frac{3}{2}}}{3\Lambda_1}} + e^{\frac{(1-4s^2)^{\frac{3}{2}}}{3\Lambda_1}} \right). \qquad (28)$$

The coefficients $C_1(s,r)$ and $C_2(s,r)$ are not equal to zero. Thus, two Fock spaces constructed with the help of the modes (20) and (21) are not equivalent and we have two different third quantized vacuum states (voids): the in-vacuum $|\, 0,\, in \,\rangle$ and out-vacuum $|\, 0,\, out \,\rangle$ (which are the states with no Friedmann Robertson Walker-like universes) defined by

$$C_{in}(s) \mid 0, in\rangle = 0 \quad \text{and} \quad C_{out}(s) \mid 0, out\,\rangle = 0, \qquad (29)$$

where $s \in \mathbf{R}$. Since the vacuum states $|\, 0,\, in \,\rangle$ and $|\, 0,\, out \,\rangle$ are not equivalent, the birth of the universes from nothing may have place, where nothing is the vacuum state $|\, 0, in \,\rangle$. The average number of universes produced from nothing, in the s-tn mode $N(s)$ is

$$N(s) = \left\langle 0, in \mid C_{out}^{\dagger}(s)\, C_{out}(s) \mid 0, in \right\rangle, \qquad (30)$$

as follows from equation (25) we get

$$N(s) = \frac{1}{2}\left(\frac{1}{2}\, e^{-\frac{(1-4s^2)^{\frac{3}{2}}}{3\Lambda_1}} - e^{\frac{(1-4s^2)^{\frac{3}{2}}}{3\Lambda_1}} \right)^2, \qquad (31)$$

considering Coleman's wormhole mechanism [8] for the vanishing cosmological constant and the constraint $\Lambda_1 \leqslant \frac{1}{8}\pi \times 10^{-120}\, m_p^4$, with $|s| \ll 1$, then the number of state $N(s)$ is

$$N(s) \approx \frac{1}{2}\, e^{\frac{2}{3\Lambda_1}(1-4s^2)^{\frac{3}{2}}}. \qquad (32)$$

This result from third quantization shows that the number of the universes produced from nothing is exponentially large.

3.2 Particular model

An interesting model derived from Bergmann Wagoner action (1) with $\omega(\phi) = \omega_0 = \text{const}$, $\lambda(\phi) = 0$ and $N = 1$ is the Brans-Dicke theory

$$S = \frac{1}{l_p^2}\int \sqrt{-g}\left[\phi R^{(4)} - \frac{\omega(\phi)}{\phi}\, g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 2\phi\lambda(\phi) \right] dt\,. \qquad (33)$$

By means of new variables

$$x = \ln(a^2\phi)\,, \qquad y = \ln\phi^{\frac{1}{\rho}}\,, \qquad dt = a d\tau\,, \qquad (34)$$

where $\rho^2 = \frac{3}{2\omega_0+3}$, action (33) transforms into

$$S = \frac{1}{2}\int \left[\frac{x'^2}{4} - \frac{y'^2}{4} - 1 \right] e^x d\tau\,, \qquad (35)$$

the WDW equation for this model is

$$\left[x^{-A}\partial_x(x^A\partial_x) - \partial_y^2 - \frac{e^{2x}}{4} \right] \psi(x,y) = 0\,, \qquad (36)$$

the ambiguity of factor ordering is encoded in the A parameter. The third quantized action to yield the WDW equation (36) is

$$S_{3Q} = \frac{1}{2}\int \left[(\partial_x\psi)^2 - (\partial_y\psi)^2 + \frac{e^{2x}}{4}\,\psi^2 \right] dx\, dy\,. \qquad (37)$$

Again, in order to quantize this toy model, we impose equal time commutation relations given by (17–19), and by means of normal mode functions ψ_p we can expand the field $\psi(x,y)$. A suitable normalized out-mode function with positive frequency for large scales, is

$$\psi_p^{out}(x,y) = \frac{1}{2\sqrt{2}}\, e^{-\frac{\pi}{2}|p|}\, H_{-q}^{(2)}\, \frac{ie^x}{2}\, e^{ipy}, \qquad (38)$$

where $H_{-q}^{(2)}$ is a Hankel function and $q = -i\,|p|$. The normalized in-mode function is

$$\psi_p^{in}(x,y) = \frac{e^{\frac{\pi}{2}|p|}}{2\sinh^{\frac{1}{2}}(\pi\,|p|)}\, J_q\, \frac{ie^x}{2}\, e^{ipy}, \qquad (39)$$

where J_q is a first class Bessel function. In the classically allowed regions the positive frequency modes correspond to the expanding universe [9]. As both wavefunctions (38) and (39) are complete, they are related to each other by a Bogoliubov transformation. The corresponding coefficients are

$$C_1(p,q) = \delta(p-q)\frac{1}{\sqrt{1-e^{-2\pi|p|}}}\,, \qquad (40)$$

and

$$C_2(p,q) = \delta(p+q)\frac{1}{\sqrt{e^{2\pi|p|}-1}}\,. \qquad (41)$$

The coefficients $C_1(p)$ and $C_2(p)$ are not equal to zero and satisfy the probability conservation condition

$$|\, C_1(p)\,|^2 - |\, C_2(p)\,|^2 = 1\,.$$

In this way, it can be constructed two not equivalent Fock spaces by means of (38) and (39). These two different third quantized vacuum states, the in-vacuum $|\,0,\,in\,\rangle$ and out-vacuum $|\,0,\,out\rangle$ are defined by (29). The average number of universes created from nothing, i.e., the in-vacuum in the p-th $N(p)$, is

$$\begin{aligned} N(p) &= \langle 0, in \mid C^{\dagger}_{out}(p)\, C_{out}(p) \mid 0, in\rangle = \\ &= |\, C_2(p)\,|^2 = \\ &= \frac{1}{e^{2\pi|p|}-1}\,. \end{aligned} \tag{42}$$

This expresion corresponds to a Planckian distribution of universes.

4 Conclusions

By means of a suitable choice of lapse function and independent variables, we have solved the WDW equation in the Bergmann-Wagoner gravitational theory for a cosmological function of the form $\lambda(\phi)=\Lambda_1\cosh[2y(\phi)]+\Lambda_2\sinh[2y(\phi)]$, this kind of cosmological term is important because of new scenario of extended inflation [10]. Also, we have studied on the third quantization of Bergmann-Wagoner and Brans-Dicke models, in which time is related by the scalar factor of universe and the space coordinate is related with the scalar field. The universe is created from stable vacuum obtained by the Bogoliubov-type transformation just as it is in the quantum field theory.

One of the main results of third quantization is that the number of universes produced from nothing is exponentially large. We calculated the number density of the universes creating fron nothing and found that the initial state $|\ 0, in\rangle$ is populated by a Planckian distribution of universes.

Acknowledgements

This work was supported by EDI-IPN and COFAA-IPN GRANTS and Research Project SIP-20082788.

Submitted on February 27, 2008
Accepted on March 29, 2008

References

1. Pervushin V. N. and Zinchuk V. A., Quantum cosmological origin of universes. arXiv: gr-qc/0504123.
2. Yasunori F. and Maeda K. I. Scalar-tensor theories of gravitation. Cambr. Univ. Press, Cambridge, England, 2002.
3. Vakili V., Jalalzadeh S. and Sepangi H. R. *Annals of Phys.*, 2006, v. 321, 2491.
4. Bergmann P. G. *Int. J. Theor. Phys.*, 1968, v. 1, 25; Wagoner R. V. *Phys. Rev. D*, 1970, v. 1, 3209.
5. Mora C. and Pimentel L. O. *Spacetime and Substance*, 2007, v. 8, 49.
6. Pervushin V. N. and Zinchuk V. A. Bogoliubov's integrals of motion in quantum cosmology and gravity. arXiv: gr-qc/0601067.
7. Pimentel L. O. and Mora C. *Phys. Lett. A*, 2001, v. 280, 191.
8. Coleman S. *Nucl. Phys. B*, 1988, v. 307, 864.
9. Horiguchi T. *Mod. Phys. Lett. A*, 1993, v. 8, 777.
10. Kim W. and Yoon M. S. *Phys. Lett. B*, 2007, v. 645, 82.