An Asymptotic Solution for the Navier-Stokes Equation

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We have used as the velocity field of a fluid the functional form derived in Casuso (2007), obtained by studying the origin of turbulence as a consequence of a new description of the density distribution of matter as a modified discontinuous Dirichlet integral. As an interesting result we have found that this functional form for velocities is a solution to the Navier-Stokes equation when considering asymptotic behaviour, i.e. for large values of time.

1 Introduction

The Euler and Navier-Stokes equations describe the motion of a fluid. These equations are to be solved for an unknown velocity vector \( \vec{u}(\vec{r}, t) \) and pressure \( P(\vec{r}, t) \), defined for position \( \vec{r} \) and time \( t \gg 0 \). We restrict attention here to incompressible fluids filling all real space. Then the Navier-Stokes equations are: a) Newton's law \( \sum \vec{f} = m \vec{a} \) for a fluid element subject to the external force \( \vec{g} \) (gravity) and to the forces arising from pressure and friction, and b) The condition of incompressibility. A fundamental problem in the analysis is to find any physically reasonable solution for the Navier-Stokes equation, and indeed to show that such a solution exists. Many numerical computations appear to exhibit blowup for solutions of the Euler equations (the same as Navier-Stokes equations) but for zero viscosity, but the extreme numerical instability of the equations makes it very hard to draw reliable conclusions (see Bertozzi and Majda 2002 [1]). Important progress has been made in understanding weak solutions of the Navier-Stokes equations (Leray 1934 [2], Khon and Nirenberg 1982 [3], Scheffer 1993 [4], Schnirelman 1997 [5], Caffarelli and Lin 1998 [6]). This type of solutions means that one integrates the equation against a test function, and then integrates by parts to make the derivatives fall on the test function. In the present paper we test directly the validity of a solution which was obtained previously from the study of turbulence.

2 Demonstration of validity of the asymptotic solution

We start from the Navier-Stokes equation for one-dimension:

\[
\frac{\partial u_k}{\partial t} + u_k \frac{\partial u_k}{\partial x} = -\frac{\partial P}{\partial x} + g, \tag{1}
\]

where \( \nu \) is a positive coefficient (viscosity) and \( g \) means a nearly constant gravitational force per unit mass (an externally applied force).

Taking from Casuso, 2007 [7], the functional form derived for the velocity of a fluid

\[
u_k = -\frac{\pi}{2} \sum_{k} \sin\left(\frac{\pi x_k}{2}\right) e^{it(k+\kappa)} + \text{const}, \tag{2}
\]

where \(-x_k \leq x + k \leq x_k\), \(k\) describe the central positions of real matter structures such as atomic nuclei and \(x_k\) means the size of these structures. Assuming a polytropic relation between pressure \(P\) and density \(\rho\) via the sound speed \(s\) we have:

\[
P = s^2 \rho = \frac{s^2}{\pi} \sum_{k} \sin\left(\frac{\pi x_k}{2}\right) \int_{t}^{t+s} e^{it(k+\kappa)} dt. \tag{3}
\]

Puting equations (2) and (3) into equation (1) we obtain:

\[
A + B = C + g, \tag{4}
\]

where

\[
A = \frac{-s^2}{\pi} \sum_{k} \frac{\cos(\pi x_k t)}{t^2} \sin(x_k t) + \sin(x_k t), \tag{5}
\]

\[
B = \frac{-s^2}{\pi} \sum_{k} \frac{\sin(\pi x_k t)}{t^2} e^{it(k+\kappa)} + \text{const}, \tag{6}
\]

\[
C = \nu \left[ \frac{-s^2}{\pi} \sum_{k} \sin(x_k t) e^{it(k+\kappa)} \right] - \frac{s^2}{\pi} \sum_{k} \int_{t}^{t+s} \sin(x_k t) e^{it(k+\kappa)} dt. \tag{7}
\]

Now taking the asymptotic approximation, at very large time \(t\), we obtain

\[
u \sin(x_k t) e^{it(k+\kappa)} = -\frac{s^2}{\pi} \sum_{k} \sin(x_k t) e^{it(k+\kappa)} dt + g, \tag{8}
\]

and differentiating and taking only the real part, we have

\[
x_k \cos(x_k t) = -\frac{s^2}{\pi \nu} \sin(x_k t), \tag{9}
\]

which is the same as

\[
-x_k \pi \nu \frac{s^2}{s^2} = \tan(x_k t) \tag{10}
\]

then, in the limiting case (real case) \(x_k \rightarrow 0\) and, again at very
large time $t$, we have the solutions
\[ x_k t = 0, \pi, 2\pi, 3\pi, \ldots, n\pi \] (11)
with $n$ being any integer number. So we have demonstrated that the equation (2) is a solution for the Navier-Stokes equation in one dimension.

Now, for the general case of 3-dimensions we have to generalize the functional form which describes the nature of matter in Casuso, 2007 [7], in the sense of taking a new form for the density
\[ \rho = \frac{1}{\pi} \sum_k \sin(r_k t) e^{it(r+k)} dt, \]
where \( \tau = \sqrt{x^2 + y^2 + z^2} \), and applying the continuity equation
\[ \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} (\rho u_x) - \frac{\partial}{\partial y} (\rho u_y) - \frac{\partial}{\partial z} (\rho u_z). \] (13)

Using the condition of incompressibility included in Navier-Stokes equations
\[ \text{div} \mathbf{u} = 0 \]
and assuming isotropy for the velocity field $u_x \sim u_y \sim u_z$, we have
\begin{align*}
  u_x &= u_y = u_z = -\frac{\tau}{\pi(x + y + z)} \\
  &\times \sum_k \sin(r_k t) \frac{\sin(r_k t)}{\tau^2} e^{it(r+k)} + \text{const}, \quad (15)
\end{align*}
where $-r_k \leq r + k \leq r_k$. Including this expression for the velocity in the 3-dimensional Navier-Stokes main equation (taking into account the condition $\text{div} \mathbf{u} = 0$)
\[ \frac{\partial}{\partial t} u_x = \nu \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] u_x - \frac{\partial P}{\partial x} + g, \] (16)
we obtain
\[ \frac{\tau}{\pi(x + y + z)} \sum_k e^{it(r+k)} \times \] \begin{align*}
  &\left[ r_k \cos(r_k t) + \frac{(r + k) \sin(r_k t)}{\tau^2} \right] \frac{\sin(r_k t)}{\tau^2} \frac{2 \sin(r_k t)}{\tau^2} + \frac{\sin(r_k t)}{\tau^2} \\
  &= \nu \Delta u_x - \frac{\partial P}{\partial x} + g, \quad (17)
\end{align*}
where $\Delta$ means $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Again taking the approximation of very large time, we have
\[ \frac{\partial P}{\partial x} = g, \] (18)
i.e.
\[ i \frac{s^2}{\tau \pi} \sum_k \sin(r_k t) e^{it(r+k)} dt = g. \] (19)
Taking the partial derivative with respect to time we obtain
\[ i \frac{s^2}{\tau \pi} \sum_k \sin(r_k t) e^{it(r+k)} = 0 \] (20)
or (which is the same),
\[ e^{it(r+k)} \sin(r_k t) = 0, \] (21)
i.e.
\[ (\cos[(r + k)t] - i \sin[(r + k)t]) \sin(r_k t) = 0. \] (22)
Taking only the real part
\[ \sin(r_k t) \cos[(r + k)t] = 0. \] (23)
So, we have two solutions: (a) $r_k t = 0, \pi, 2\pi, \ldots, n\pi$, and (b) $(r + k)t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, (2n + 1)\frac{\pi}{2}$. We must note that the solution (a) is similar to the 1-dimension solution.

3 Conclusions

By using a new discontinuous functional form for matter density distribution, derived from consideration of the origin of turbulence, we have found an asymptotic solution to the Navier-Stokes equation for the three dimensional case. This result, while of intrinsic interest, may point towards new ways of deriving a general solution.

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