Relativistic Mechanics in Gravitational Fields Exterior to Rotating Homogeneous Mass Distributions within Regions of Spherical Geometry

Chifu Ebenezer Ndikilar*, Samuel Xede Kofi Howusu†, and Lucas Williams Lumbi‡

*Physics Department, Gombe State University, P.M.B. 127, Gombe, Gombe State, Nigeria
E-mail: ebenechifu@yahoo.com

†Physics Department, Kogi State University, Anyighba, Kogi State, Nigeria
E-mail: sxkhowusu@yahoo.co.uk

‡Physics Department, Nasarawa State University, Keffi, Nasarawa State, Nigeria
E-mail: williamslucas66@yahoo.com

General Relativistic metric tensors for gravitational fields exterior to homogeneous spherical mass distributions rotating with constant angular velocity about a fixed diameter are constructed. The coefficients of affine connection for the gravitational field are used to derive equations of motion for test particles. The laws of conservation of energy and angular momentum are deduced using the generalized Lagrangian. The law of conservation of angular momentum is found to be equal to that in Schwarzschild's gravitational field. The planetary equation of motion and the equation of motion for a photon in the vicinity of the rotating spherical mass distribution have rotational terms not found in Schwarzschild's field.

1 Introduction

General Relativity is the geometrical theory of gravitation published by Albert Einstein in 1915/1916 [1–3]. It unifies Special Relativity and Sir Isaac Newton's law of universal gravitation with the insight that gravitation is not due to a force but rather a manifestation of curved space and time, with the curvature being produced by the mass-energy and momentum content of the space time. After the publication of Einstein's geometrical field equations in 1915, the search for their exact and analytical solutions for all the gravitational fields in nature began [3].

The first method of approach to the construction of exact analytical solutions of Einstein's geometrical gravitational field equations was to find a mapping under which the metric tensor assumed a simple form, such as the vanishing of the off-diagonal elements. This method led to the first analytical solution — the famous Schwarzschild's solution [3]. The second method was to assume that the metric tensor contains symmetries — assumed forms of the associated Killing vectors. The assumption of axially asymmetric metric tensor led to the solution found by Weyl and Levi-Civita [4–11]. The fourth method was to seek Taylor series expansion of some initial value hyper surface, subject to consistent initial value data. This method has not proved successful in generating solutions [4–11].

We now introduce our method and approach to the construction of exact analytical solutions of Einstein's geometrical gravitational field equations [12, 13] as an extension of Schwarzschild analytical solution of Einstein's gravitational field equations. Schwarzschild's metric is well known to be the metric due to a static spherically symmetric body situated in empty space such as the Sun or a star [3, 12, 13]. Schwarzschild's metric is well known to be given as

$$g_{00} = 1 - \frac{2GM}{c^2 r}\,, \tag{1.1}$$

$$g_{11} = -\left[1 - \frac{2GM}{c^2 r}\right]^{-1}, \tag{1.2}$$

$$g_{22} = -r^2, \tag{1.3}$$

$$g_{33} = -r^2 \sin^2\theta\,, \tag{1.4}$$

$$g_{\mu\nu} = 0 \ \text{ otherwise}, \tag{1.5}$$

where $r > R$, the radius of the static spherical mass, G is the universal gravitational constant, M is the total mass of the distribution and c is the speed of light in vacuum. It can be easily recognized [12, 13] that the above metric can be written as

$$g_{00} = 1 + \frac{2f(r)}{c^2}\,, \tag{1.6}$$

$$g_{11} = -\left[1 + \frac{2f(r)}{c^2}\right]^{-1}, \tag{1.7}$$

$$g_{22} = -r^2, \tag{1.8}$$

$$g_{33} = -r^2 \sin^2\theta\,, \tag{1.9}$$

$$g_{\mu\nu} = 0 \ \text{ otherwise}, \tag{1.10}$$

where

$$f(r) = -\frac{GM}{r}\,. \tag{1.11}$$

We thus deduce that generally, $f(r)$ is an arbitrary function determined by the distribution. In this case, it is a function of the radial coordinate r only; since the distribution and hence its exterior gravitational field possess spherical symmetry. From the condition that these metric components should reduce to the field of a point mass located at the origin and contain Newton's equations of motion in the field of the spherical body, it follows that generally, $f(r)$ is approximately equal to the Newtonian gravitational scalar potential in the exterior region of the body, $\Phi(r)$ [12, 13].

Hence, we postulate that the arbitrary function f is solely determined by the mass or pressure distribution and hence possesses all the symmetries of the latter, a priori. Thus, by substituting the generalized arbitrary function possessing all the symmetries of the distribution in to Einstein's gravitational field equations in spherical polar coordinates, explicit equations satisfied by the single arbitrary function, $f(t, r, \theta, \phi)$, can be obtained. These equations can then be integrated exactly to obtain the exact expressions for the arbitrary function. Also, a sound and satisfactory approximate expression can be obtained from the well known fact of General Relativity [12, 13] that in the gravitational field of any distribution of mass;

$$g_{00} \approx 1 + \frac{2}{c^2}\,\Phi(t, r, \theta, \phi)\,. \tag{1.12}$$

It therefore follows that:

$$f(t, r, \theta, \phi) \approx \Phi(t, r, \theta, \phi)\,. \tag{1.13}$$

In a recent article [13], we studied spherical mass distributions in which the material inside the sphere experiences a spherically symmetric radial displacement. In this article, we now study general relativistic mechanics in gravitational fields produced by homogeneous mass distributions rotating with constant angular velocity about a fixed diameter within a static sphere placed in empty space.

2 Coefficients of affine connection

Consider a static sphere of total mass M and density ρ. Also, suppose the mass or pressure distribution within the sphere is homogeneous and rotating with uniform angular velocity about a fixed diameter. More concisely, suppose we have a static spherical object filled with a gas say and the gas is made to rotate with a constant velocity about a fixed diameter. In otherwords, the material inside the sphere is rotating uniformly but the sphere is static. Such a mass distribution might be hypothetical or exist physically or exist astrophysically. For this mass distribution, it is eminent that our arbitrary function will be independent of the coordinate time and azimuthal angle. Thus, the covariant metric for this gravitational field is given as

$$g_{00} = 1 + \frac{2f(r,\theta)}{c^2}\,, \tag{2.1}$$

$$g_{11} = -\left[1 + \frac{2f(r,\theta)}{c^2}\right]^{-1}, \tag{2.2}$$

$$g_{22} = -r^2, \tag{2.3}$$

$$g_{33} = -r^2\sin^2\theta\,, \tag{2.4}$$

$$g_{\mu\nu} = 0 \;\text{ otherwise}, \tag{2.5}$$

where $f(r,\theta)$ is an arbitrary function determined by the mass distribution within the sphere. It is instructive to note that our generalized metric tensor satisfy Einstein's field equations and the invariance of the line element; by virtue of their construction [1, 12]. An outstanding theoretical and astrophysical consequence of this metric tensor is that the resultant Einstein's field equations have only one unknown function, $f(r,\theta)$. Solutions to these field equations give explicit expressions for the function $f(r,\theta)$. In approximate gravitational fields, $f(r,\theta)$ can be conveniently equated to the gravitational scalar potential exterior to the homogeneous spherical mass distribution [1, 12–14]. It is most interesting and instructive to note that the rotation of the homogeneous mass distribution within the static sphere about a fixed diameter is taken care of by polar angle, θ in the function $f(r,\theta)$. Also, if the sphere is made to rotate about a fixed diameter, there will be additional off diagonal components to the metric tensor. Thus, in this analysis, the static nature of the sphere results in the vanishing of the off diagonal components of the metric.

The contravariant metric tensor for the gravitational field, obtained using the Quotient Theorem of tensor analysis [15] is given as

$$g^{00} = \left[1 + \frac{2f(r,\theta)}{c^2}\right]^{-1}, \tag{2.6}$$

$$g^{11} = -\left[1 + \frac{2f(r,\theta)}{c^2}\right], \tag{2.7}$$

$$g^{22} = -r^{-2}, \tag{2.8}$$

$$g^{33} = -\left(r^2\sin^2\theta\right)^{-1}, \tag{2.9}$$

$$g^{\mu\nu} = 0 \;\text{ otherwise}, \tag{2.10}$$

It is well known that the coefficients of affine connection for any gravitational field are defined in terms of the metric tensor [14, 15] as;

$$\Gamma^{\sigma}_{\mu\lambda} = \frac{1}{2}\,g^{\sigma\nu}\left(g_{\mu\nu,\lambda} + g_{\nu\lambda,\mu} - g_{\mu\lambda,\nu}\right), \tag{2.11}$$

$$\ddot{r} + \left[1 + \frac{2}{c^2} f(r,\theta)\right] \frac{\partial f(r,\theta)}{\partial r} \dot{t}^2 - \frac{1}{c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \frac{\partial f(r,\theta)}{\partial r} \dot{r}^2 - \frac{2}{c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \frac{\partial f(r,\theta)}{\partial \theta} \dot{r}\,\dot{\theta} - \\ - r \left[1 + \frac{2}{c^2} f(r,\theta)\right] \dot{\theta}^2 - r \sin^2\theta \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-2} \frac{\partial f(r,\theta)}{\partial \theta} \dot{\phi}^2 = 0 \tag{3.5}$$

where the comma as in usual notation designates partial differentiation with respect to x^λ, x^μ and x^ν. Thus, we construct the explicit expressions for the coefficients of affine connection in this gravitational field as;

$$\Gamma^0_{01} \equiv \Gamma^0_{10} = \frac{1}{c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \frac{\partial f(r,\theta)}{\partial r}, \tag{2.12}$$

$$\Gamma^0_{02} \equiv \Gamma^0_{20} = \frac{1}{c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \frac{\partial f(r,\theta)}{\partial \theta}, \tag{2.13}$$

$$\Gamma^1_{00} = \frac{1}{c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right] \frac{\partial f(r,\theta)}{\partial r}, \tag{2.14}$$

$$\Gamma^1_{11} = -\frac{1}{c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \frac{\partial f(r,\theta)}{\partial r}, \tag{2.15}$$

$$\Gamma^1_{12} \equiv \Gamma^1_{21} = -\frac{1}{c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \frac{\partial f(r,\theta)}{\partial \theta}, \tag{2.16}$$

$$\Gamma^1_{22} = -r \left[1 + \frac{2}{c^2} f(r,\theta)\right], \tag{2.17}$$

$$\Gamma^1_{33} = -r \sin^2\theta \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-2} \frac{\partial f(r,\theta)}{\partial \theta}, \tag{2.18}$$

$$\Gamma^2_{00} = \frac{1}{r^2 c^2} \frac{\partial f(r,\theta)}{\partial \theta}, \tag{2.19}$$

$$\Gamma^2_{11} = \frac{1}{r^2 c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-2} \frac{\partial f(r,\theta)}{\partial \theta}, \tag{2.20}$$

$$\Gamma^2_{12} \equiv \Gamma^2_{21} \equiv \Gamma^3_{13} \equiv \Gamma^3_{31} = -\frac{1}{r}, \tag{2.21}$$

$$\Gamma^2_{33} = -\frac{1}{2} \sin 2\theta, \tag{2.22}$$

$$\Gamma^3_{23} \equiv \Gamma^3_{32} = \cot\theta, \tag{2.23}$$

$$\Gamma^\sigma_{\mu\lambda} = 0 \ \text{ otherwise}, \tag{2.24}$$

Thus, the gravitational field exterior to a homogeneous rotating mass distribution within regions of spherical geometry has twelve distinct non zero affine connection coefficients. These coefficients are very instrumental in the construction of general relativistic equations of motion for particles of non-zero rest mass.

3 Motion of test particles

A test mass is one which is so small that the gravitational field produced by it is so negligible that it doesn't have any effect on the space metric. A test mass is a continuous body, which is approximated by its geometrical centre; it has nothing in common with a point mass whose density should obviously be infinite [16].

The general relativistic equation of motion for particles of non-zero rest masses is given [1, 12–14, 17] as

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \left(\frac{dx^\nu}{d\tau}\right) \left(\frac{dx^\lambda}{d\tau}\right) = 0, \tag{3.1}$$

where τ is the proper time. To construct the equations of motion for test particles, we proceed as follows

Setting $\mu = 0$ in equation (3.1) and substituting equations (2.12) and (2.13) gives the time equation of motion as

$$\ddot{t} + \frac{2}{c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \frac{\partial f(r,\theta)}{\partial r} \dot{t}\,\dot{r} + \\ + \frac{2}{c^2} \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \frac{\partial f(r,\theta)}{\partial \theta} \dot{t}\,\dot{\theta} = 0, \tag{3.2}$$

where the dot denotes differentiation with respect to proper time. Equation (3.2) is the time equation of motion for particles of non-zero rest masses in this gravitational field. It reduces to Schwarzschild's time equation when $f(r,\theta)$ reduces to $f(r)$. The third term in equation (3.2) is the contribution of the rotation of the mass within the sphere; it does not appear in Schwarzschild's time equation of motion for test particles [1, 12–14, 17]. It is interesting and instructive to realize that equation (3.2) can be written equally as

$$\frac{d}{d\tau} \left(\ln \dot{t}\right) + \frac{d}{d\tau} \left[\ln \left(1 + \frac{2}{c^2} f(r,\theta)\right)\right] = 0. \tag{3.3}$$

Integrating equation (3.3) yields

$$\dot{t} = A \left(1 + \frac{2}{c^2} f(r,\theta)\right)^{-1}, \tag{3.4}$$

where A is the constant of integration (as $t \to \tau$, $f(r,\theta) \to 0$ and thus the constant A is equivalent to unity). Equation (3.4) is the expression for the variation of the time on a clock moving in this gravitational field. It is of same form as that in Schwarzschild's gravitational field [1, 12–14, 17].

Similarly, setting $\mu = 1$ in equation (3.1) gives the radial equation of motion as formula (3.5) on the top of this page.

For pure radial motion $\dot\theta \equiv \dot\phi = 0$ and hence equation (3.5) reduces to

$$\ddot r + \left[1 + \frac{2}{c^2}\, f(r,\theta)\right]^{-1} \frac{\partial f(r,\theta)}{\partial r} \left(1 - \frac{1}{c^2}\,\dot r^2\right) = 0\,. \quad (3.6)$$

The instantaneous speed of a particle of non-zero rest mass in this gravitational field can be obtained from equations (3.5) and (3.6).

Also, setting $\mu = 2$ and $\mu = 3$ in equation (3.1) gives the respective polar and azimuthal equations of motion as

$$\ddot\theta + \frac{1}{r^2}\frac{\partial f(r,\theta)}{\partial\theta}\,\dot t^2 + \frac{1}{r^2c^2}\left[1 + \frac{2}{c^2}\, f(r,\theta)\right]^{-2} \times \times \frac{\partial f(r,\theta)}{\partial\theta}\,\dot r^2 + \frac{2}{r}\,\dot r\,\dot\theta - \frac{1}{2}\,(\dot\phi)^2 \sin 2\theta = 0 \quad (3.7)$$

and

$$\ddot\phi + \frac{2}{r}\,\dot r\,\dot\phi + 2\dot\theta\,\dot\phi \cot\theta = 0\,. \quad (3.8)$$

It is instructive to note that equation (3.7) reduces satisfactorily to the polar equation of motion in Schwarzschild's gravitational field when $f(r,\theta)$ reduces to $f(r)$. Equation (3.8) is equal to the azimuthal equation of motion for particles of non-zero rest masses in Schwarzschild's field. Thus, the instantaneous azimuthal angular velocity from our field is exactly the same as that obtained from Newton's theory of gravitation [14] and Schwarzschild's metric [1, 12, 13, 17].

4 Orbits

The Lagrangian in the space time exterior to any mass or pressure distribution is defined as [17]

$$L = \frac{1}{c}\left(-\,g_{\alpha\beta}\,\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right)^{\frac{1}{2}} = 0\,. \quad (4.1)$$

Thus, in our gravitational field, the Lagrangian can be written as

$$L = \frac{1}{c}\left[-\,g_{00}\left(\frac{dt}{d\tau}\right)^2 - g_{11}\left(\frac{dr}{d\tau}\right)^2\right]^{\frac{1}{2}} - -\frac{1}{c}\left[g_{22}\left(\frac{d\theta}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2\right]^{\frac{1}{2}} = 0\,. \quad (4.2)$$

Considering motion confined to the equatorial plane of the homogeneous spherical body, $\theta = \frac{\pi}{2}$ and hence $d\theta = 0$. Thus, in the equatorial plane, equation (4.2) reduces to

$$L = \frac{1}{c}\left[-\,g_{00}\left(\frac{dt}{d\tau}\right)^2 - -\,g_{11}\left(\frac{dr}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2\right]^{\frac{1}{2}} = 0\,. \quad (4.3)$$

Substituting the explicit expressions for the components of the metric tensor in the equatorial plane of the spherical body yields

$$L = \frac{1}{c}\left[-\left(1 + \frac{2}{c^2}\, f(r,\theta)\right)\dot t^2\right]^{\frac{1}{2}} + + \frac{1}{c}\left[\left(1 + \frac{2}{c^2}\, f(r,\theta)\right)^{-1}\dot r^2 + r^2\dot\phi^2\right]^{\frac{1}{2}}, \quad (4.4)$$

where the dot as in usual notation denotes differentiation with respect to proper time.

It is well known that the gravitational field is a conservative field. The Euler-lagrange equations of motion for a conservative system in which the potential energy is independent of the generalized velocities is written as [17]

$$\frac{\partial L}{\partial x^\alpha} = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot x^\alpha}\right), \quad (4.5)$$

but

$$\frac{\partial L}{\partial x^0} \equiv \frac{\partial L}{\partial t} = 0\,, \quad (4.6)$$

by the time homogeneity of the field and thus from equation (4.5), we deduce that

$$\frac{\partial L}{\partial \dot t} = \text{constant}. \quad (4.7)$$

From equation (4.4), it can be shown using equation (4.7) that

$$\left(1 + \frac{2}{c^2}\, f(r,\theta)\right)\dot t = k\,, \qquad \dot k = 0 \quad (4.8)$$

where k is a constant. This the law of conservation of energy in the equatorial plane of the gravitational field [17]. It is of same form as that in Schwarzschild's field. Also, the Lagrangian for this gravitational field is invariant to azimuthal angular rotation (space is isotropic) and hence angular momentum is conserved, thus

$$\frac{\partial L}{\partial \phi} = 0\,, \quad (4.9)$$

and from Lagrange's equation of motion and equation (4.4) it can be shown that

$$r^2\dot\phi = l\,, \qquad \dot l = 0\,, \quad (4.10)$$

where l is a constant. This is the law of conservation of angular momentum in the equatorial plane of our gravitational field. It is equivalent to that obtained in Schwarzschild's gravitational field. Thus, we deduce that the laws of conservation of total energy and angular momentum are invariant in form in the two gravitational fields.

To describe orbits in Schwarzschild's space time, the Lagrangian for permanent orbits in the equatorial plane [17] is given as;

$$L = \left\{ \left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{dt}{d\tau}\right)^2 - \right.$$
$$\left. - \frac{1}{c^2}\left[\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right]\right\}^{\frac{1}{2}}. \tag{4.11}$$

For time-like orbits, the Lagrangian gives the planetary equation of motion in Schwarzschild's space time as

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + 3\frac{GM}{c^2}\,u^2, \tag{4.12}$$

where $u = \frac{1}{r}$ and h is a constant of motion. The solution to equation (4.12) depicts the famous perihelion precession of planetary orbits [1, 14, 17]. For null orbits, the equation of motion of a photon in the vicinity of a massive sphere in Schwarzschild's field is obtained as

$$\frac{d^2u}{d\phi^2} + u = 3\,\frac{GM}{c^2}\,u^2. \tag{4.13}$$

A satisfactory theoretical explanation for the deflection of light in the vicinity of a massive sphere in Schwarzschild's space time is obtained from the solution of equation (4.13).

It is well known [17] that the Lagrangian $L = \epsilon$, with $\epsilon = 1$ for time like orbits and $\epsilon = 0$ for null orbits. Setting $L = \epsilon$ in equation (4.4) and squaring yields the Lagrangian in the equatorial plane of the gravitational field exterior to a rotating mass distribution within regions of spherical geometry as

$$\epsilon^2 = \frac{1}{c^2}\left[-\left(1 + \frac{2}{c^2}\,f(r,\theta)\right)\dot{t}^2\right] +$$
$$+ \frac{1}{c^2}\left[\left(1 + \frac{2}{c^2}\,f(r,\theta)\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right]. \tag{4.14}$$

Substituting equations (4.8) and (4.10) into equation (4.14) and simplifying yields

$$\dot{r}^2 + \left(1 + \frac{2}{c^2}\,f(r,\theta)\right)\frac{l^2}{r^2} -$$
$$- 2\epsilon^2 f(r,\theta) = c^2\epsilon^2 + k^2. \tag{4.15}$$

In most applications of general relativity, we are more interested in the shape of orbits (that is, as a function of the azimuthal angle) than in their time history [1, 14, 17]. Hence, it is instructive to transform equation (4.15) into an equation in terms of the azimuthal angle ϕ. Now, let us consider the following standard transformation

$$r = r(\phi) \quad \text{and} \quad u(\phi) = \frac{1}{r(\phi)}\,, \tag{4.16}$$

then

$$\dot{r} = -\,\frac{l}{1+u^2}\frac{du}{d\phi}\,. \tag{4.17}$$

Imposing the transformation equations (4.16) and (4.17) on (4.15) and simplifying yields

$$\left(\frac{l}{1+u^2}\frac{dt}{d\tau}\right)^2 + \left(1 + \frac{2}{c^2}\,f(u,\theta)\right)u^2 -$$
$$- 2\epsilon^2\,\frac{f(u,\theta)}{l^2} = \frac{c^2\epsilon^2 + k^2}{l^2}\,. \tag{4.18}$$

Equation (4.18) can be integrated immediately, but it leads to elliptical integrals, which are awkward to handle [14]. We thus differentiate this equation to obtain:

$$\frac{d^2u}{d\phi^2} - 2u\left(1+u^2\right)\frac{du}{d\phi} + u\left(1+u^2\right)^2 \times$$
$$\times \left(1 + \frac{2}{c^2}\,f(u,\theta)\right) = \left(\frac{2\epsilon^2}{l^2} - \frac{u^2}{c^2}\right)\left(1+u^2\right)^2\frac{\partial f}{\partial u}\,. \tag{4.19}$$

For time like orbits, equation (4.19) reduces to;

$$\frac{d^2u}{d\phi^2} - 2u\left(1+u^2\right)\frac{du}{d\phi} + u\left(1+u^2\right)^2 \times$$
$$\times \left(1 + \frac{2}{c^2}\,f(u,\theta)\right) = \left(\frac{2}{l^2} - \frac{u^2}{c^2}\right)\left(1+u^2\right)^2\frac{\partial f}{\partial u}\,. \tag{4.20}$$

This is the planetary equation of motion in the equatorial plane of this gravitational field. It can be solved to obtain the perihelion precision of planetary orbits. This equation has additional terms (resulting from the rotation of the mass distribution), not found in the corresponding equation in Schwarzschild's field. Light rays travel on null geodesics and thus equation (4.19) yields;

$$\frac{d^2u}{d\phi^2} - 2u\left(1+u^2\right)\frac{du}{d\phi} + u\left(1+u^2\right)^2 \times$$
$$\times \left(1 + \frac{2}{c^2}\,f(u,\theta)\right) = -\frac{u^2}{c^2}\left(1+u^2\right)^2\frac{\partial f}{\partial u}\,. \tag{4.21}$$

as the photon equation of motion in the vicinity of the homogeneous rotating mass distribution within a static sphere. The equation contains additional terms not found in the corresponding equation in Schwarzschild's field. In the limit of special relativity, some terms in equation (4.21) vanish and the equation becomes

$$\frac{d^2u}{d\phi^2} - 2u\left(1+u^2\right)\frac{du}{d\phi} + u\left(1+u^2\right)^2 = 0\,. \tag{4.22}$$

The solution of the special relativistic equation, (4.22), can be used to solve the general relativistic equation, (4.21). This can be done by taking the general solution of equation (4.21) to be a perturbation of the solution of equation (4.22). The immediate consequence of this analysis is that it will produce an expression for the total deflection of light grazing the massive sphere.

5 Conclusion

The equations of motion for test particles in the gravitational field exterior to a homogeneous rotating mass distribution within a static sphere were obtained as equations (3.2), (3.5), (3.7) and (3.8). Expressions for the conservation of energy and angular momentum were obtained as equations (4.8) and (4.10) respectively. The planetary equation of motion and the photon equation of motion in the vicinity of the mass where obtained as equations (4.19) and (4.20). The immediate theoretical, physical and astrophysical consequences of the results obtained in this article are three fold.

Firstly, the planetary equation of motion and the photon equation have additional rotational terms not found in Schwarzschild's gravitational field. These equations are opened up for further research work and astrophysical interpretations.

Secondly, in approximate gravitational fields, the arbitrary function $f(r,\theta)$ can be conveniently equated to the gravitational scalar potential exterior to the body. Thus, in approximate fields, the complete solutions for the derived equations of motion can be constructed.

Thirdly, Einstein's field equations constructed using our metric tensor have only one unknown function, $f(r,\theta)$. Solution to these field equations give explicit expressions for the function, $f(r,\theta)$, which can then be interpreted physically and used in our equations of motion. Thus, our method places Einstein's geometrical gravitational field theory on the same footing with Newton's dynamical gravitational field theory; as our method introduces the dependence of the field on one and only one dependent variable, $f(r,\theta)$, comparable to one and only one gravitational scalar potential function in Newton's theory [12, 13].

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References

1. Bergmann P. G. Introduction to the theory of relativity. Prentice Hall, New Delhi, 1987.
2. Einstein A. The foundation of the General Theory of Relativity. *Annalen der Physik*, 1916, Bd. 49, 12–34.
3. Schwarzschild K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, 189–196 (published in English as: Schwarzschild K. On the gravitational field of a point mass according to Einstein's theory. *Abraham Zelmanov Journal*, 2008, v. 1, 10–19).
4. Finster F., et al. Decay of solutions of the wave equation in the Kerr geometry. *Communications in Mathematical Physics*, 2006, v. 264, 465–503.
5. Anderson L., et al. Assymptotic silence of generic cosmological singularities. *Physical Review Letters*, 2001, v. 94, 51–101.
6. Czerniawski J. What is wrong with Schwarzschild's coordinates. *Concepts of Physics*, 2006, v. 3, 309–320.
7. MacCallum H. Finding and using exact solutions of the Einstein equation. arXiv: 0314.4133.
8. Rendall M. Local and global existence theorems for the Einstein equations. *Living Reviews in Relativity*, 2005; arXiv: 1092.31.
9. Stephani H., et al. Exact solutions of Einstein's field equations. Cambridge Monographs Publ., London, 2003.
10. Friedrich H. On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asmptotic structure. *Communications in Mathematical Physics*, 1986, v. 107, 587–609.
11. Berger B., et al. Oscillatory approach to the singularity in vacuum spacetimes with T^2 Isometry. *Physical Reviews D*, 2001, v. 64, 6–20.
12. Howusu S.X.K. The 210 astrophysical solutions plus 210 cosmological solutions of Einstein's geometrical gravitational field equations. Jos University Press, Jos, 2007 (also available on http://www.natphilweb.com).
13. Chifu E.N. and Howusu S.X.K. Gravitational radiation and propagation field equation exterior to astrophysically real or hypothetical time varying distributions of mass within regions of spherical geometry. *Physics Essays*, 2009, v. 22, no. 1, 73–77.
14. Weinberg S. Gravitation and cosmology, J. Wiley & Sons, New York, 1972.
15. Arfken G. Mathematical methods for physicists. Academic Press, New York, 1995.
16. Rabounski D. and Borissova L. Reply to the "Certain Conceptual Anomalies in Einstein's Theory of Relativity" and related questions. *Progress in Physics*. 2008, v. 2, 166–168.
17. Dunsby P. An introduction to tensors and relativity. Shiva, Cape Town, 2000.