

# A Derivation of $\pi(n)$ Based on a Stability Analysis of the Riemann-Zeta Function

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The prime-number counting function  $\pi(n)$ , which is significant in the prime number theorem, is derived by analyzing the region of convergence of the real-part of the Riemann-Zeta function using the unilateral  $z$ -transform. In order to satisfy the stability criteria of the  $z$ -transform, it is found that the real part of the Riemann-Zeta function must converge to the prime-counting function.

## 1 Introduction

The Riemann-Zeta function, which is an infinite series in a complex variable  $s$ , has been shown to be useful in analyzing nuclear energy levels [1] and the filling of  $s_1$ -shell electrons in the periodic table [2]. The following analysis of the Riemann-Zeta function with a  $z$ -transform shows the stability zones and requirements for the real and complex variables.

## 2 Stability with the $z$ -transform

The Riemann-Zeta function is defined as

$$\Gamma(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1}$$

We start by setting the following equality

$$\Gamma(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} e^{-as}. \tag{2}$$

Then by simplifying

$$n^{-s} = e^{-as} = e^{-a(r+j\omega)} \tag{3}$$

and taking natural logarithm of both sides we obtain

$$-s \ln(n) = -as. \tag{4}$$

We then find the constant  $a$  such that

$$a = \ln(n). \tag{5}$$

We then apply the unilateral  $z$ -transform on (1):

$$\Gamma(s) = \sum_{n=1}^{\infty} n^{-s} z^{-n} = \sum_{n=1}^{\infty} e^{-as} z^{-n} = \sum_{n=1}^{\infty} e^{-a(r+j\omega)} z^{-n}. \tag{6}$$

Substituting (5), the real part of (6) becomes:

$$\text{Re} [\Gamma(s)] = \sum_{n=1}^{\infty} e^{-ar} z^{-n} = \sum_{n=1}^{\infty} e^{-r \ln(n)} z^{-n}. \tag{7}$$

In order to find the region of convergence (ROC) of (7), we have to factor (7) to the common exponent  $-n$ , which requires

$$r = n / \ln(n), \tag{8}$$

which is the same as saying that the real part of  $\Gamma(s)$  must converge to the prime-number counting function  $\pi(n)$ . With (8) satisfied, (7) becomes

$$\text{Re} [\Gamma(s)] = \sum_{n=1}^{\infty} (ez)^{-n}. \tag{9}$$

which has a region of convergence (ROC)

$$\text{ROC} = \frac{1}{1 - \frac{1}{ez}}. \tag{10}$$

To be within the region of convergence,  $z$  must satisfy the following relation

$$|z| > e^{-1} \quad \text{or} \quad |z| > 0.368. \tag{11}$$

which, places  $z$  within the critical strip. It can also be shown that the imaginary part of (6)

$$\text{Im} [\Gamma(s)] = \sum_{n=1}^{\infty} e^{-aj\omega} z^{-n} = \sum_{n=1}^{\infty} e^{-j\omega \ln(n)} z^{-n}. \tag{12}$$

converges based on the Fourier series of  $\sum e^{-j\omega \ln(n)}$ .

## 3 Conclusions

The prime number-counting function  $\pi(n)$  has been derived from a stability analysis of the Riemann-Zeta function using the  $z$ -transform. It is found that the real part of the roots of the zeta function correspond to  $\pi(n)$  under the conditions of stability dictated by the unit-circle of the  $z$ -transform. The distribution of prime numbers has been found to be useful in analyzing electron and nuclear energy levels.

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## References

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2. Harney M. The stability of electron orbital shells based on a model of the Riemann-Zeta function. *Progress in Physics*, 2008, v. 1, 1–5.