Black Holes in the Framework of the Metric Tensor Exterior to the Sun and Planets

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The conditions for the Sun and oblate spheroidal planets in the solar system to reduce to black holes is investigated. The metric tensor exterior to oblate spheroidal masses indicates that for the Sun to reduce to a black hole, its mass must condense by a factor of 2.32250×10^5. Using Schwarzschild's metric, this factor is obtained as 2.3649×10^5. Similar results are obtained for oblate spheroidal planets in the solar system.

1 Introduction

It is well known that whenever an object becomes sufficiently compact, general relativity predicts the formation of a black hole: a region of space from which nothing, not even light can escape. The collapse of any mass to the Schwarzschild radius appears to an outside observer to take an infinite time and the events at distances beyond this radius are unobservable from outside, thus the name black hole. From an astronomical point of view, the most important property of compact objects such as black holes is that they provide a superbly efficient mechanism for converting gravitational energy into radiation [1].

The world line element in Schwarzschild's field is well known to be given by [1]

$$c^2d\tau^2 = c^2\left[1-\frac{2GM}{c^2r}\right]dt^2-\left[1-\frac{2GM}{c^2r}\right]^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{1}$$

This metric has a singularity, (denoted by r_s) called the Schwarzschild singularity (or radius) at

$$r_s = \frac{2GM}{c^2}. \tag{2}$$

For most physical bodies in the universe, the Schwarzschild radius is much smaller than the radius of their surfaces. Hence for most bodies, there does not exist a Schwarzschild singularity. It is however, speculated that there exist some bodies in the universe with the Schwarzschild radius in the exterior region. Such bodies are called black holes [1].

In this article, the factor by which the radius of the Sun and oblate spheroidal planets is reduced to form a black hole is computed using the oblate spheroidal space-time metric. The results are compared to those obtained using Schwarzschild's metric.

2 Oblate Spheroidal Space-Time Metric

It has been established [2] that the covariant metric tensor in the region exterior to a static homogeneous oblate spheroid in oblate spheroidal coordinates is given as

$$g_{00} = \left(1+\frac{2}{c^2}f(\eta,\xi)\right) \tag{3}$$

$$g_{11} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\eta^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}+\frac{\xi^2(1+\xi^2)}{(1-\eta^2)}\right] \tag{4}$$

$$g_{12}\equiv g_{21} = -\frac{a^2\eta\xi}{1+\xi^2-\eta^2}\left[1-\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right] \tag{5}$$

$$g_{22} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}+\frac{\eta^2(1-\eta^2)}{(1+\xi^2)}\right] \tag{6}$$

$$g_{33} = -a^2(1+\xi^2)(1-\eta^2) \tag{7}$$

$$g_{\mu\nu} = 0;\ \text{otherwise}. \tag{8}$$

Thus, the world line element in this field can be written as

$$c^2d\tau^2 = c^2g_{00}dt^2-g_{11}d\eta^2-2g_{12}d\eta d\xi-g_{22}d\xi^2-g_{33}d\phi^2. \tag{9}$$

Multiplying equation (9) all through by $\left(\frac{1}{dt}\right)^2$ yields

$$c^2\left(\frac{d\tau}{dt}\right)^2 = c^2g_{00}-g_{11}\left(\frac{d\eta}{dt}\right)^2-2g_{12}\frac{d\eta}{dt}\frac{d\xi}{dt} - g_{22}\left(\frac{d\xi}{dt}\right)^2-g_{33}\left(\frac{d\phi}{dt}\right)^2. \tag{10}$$

It can be concluded that the space velocity(v_s) is given as

$$v_s = g_{11}\left(\frac{d\eta}{dt}\right)^2+2g_{12}\frac{d\eta}{dt}\frac{d\xi}{dt}+g_{22}\left(\frac{d\xi}{dt}\right)^2+g_{33}\left(\frac{d\phi}{dt}\right)^2, \tag{11}$$

and the velocity of local time

$$v_\tau = c\frac{d\tau}{dt}. \tag{12}$$

The gravitational velocity can equally be defined with the aid of equation (3) as

$$v_G = \sqrt{-2f(\eta, \xi)}. \tag{13}$$

This implies that

$$c^2 = v_\tau^2 + v_G^2 + v_s^2 \tag{14}$$

or

$$c = \left|\vec{v}_\tau^2 + \vec{v}_G^2 + \vec{v}_s^2\right|. \tag{15}$$

3 Black holes in oblate spheroidal space time of Sun and planets

In the absence of gravity and acceleration, $f(\eta, \xi) = 0$ and thus $v_G = 0$. Hence, v_s can be written explicitly as

$$\begin{aligned} v_s^2 = &\left[\frac{a^2\eta^2}{1+\xi^2-\eta^2} + \frac{\xi^2\left(1+\xi^2\right)}{1-\eta^2}\right]\left(\frac{d\eta}{dt}\right)^2 \\ &+\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2 + \frac{\eta^2(1-\eta^2)}{1+\xi^2}\right]\left(\frac{d\xi}{dt}\right)^2 \\ &+a^2\left(1+\xi^2\right)\left(1-\eta^2\right)\left(\frac{d\phi}{dt}\right)^2. \end{aligned} \tag{16}$$

Thus in the absence of gravity, equation(14) reduces to

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{v_s^2}{c^2}}. \tag{17}$$

It can basically be seen that equation(15) establishes a maximum value of c and hence the gravitational velocity v_G can never exceed c. An approximate expression for $f(\eta, \xi)$ along the equator of an oblate spheroid [3] is

$$f(\eta, \xi) \approx \frac{B_0}{3\xi^2}\left(1 + 3\xi^2\right)i + \frac{B_2}{30\xi^3}\left(7 + 15\xi^2\right)i \tag{18}$$

where B_0 and B_2 are constants. Equation(18) can be written equally as

$$f(\eta, \xi) \approx -\left(\frac{C}{\xi} + \frac{D}{\xi^3}\right) \tag{19}$$

where C and D are equally constants. These constants can easily be computed for the oblate spheroidal astrophysical bodies in the solar system and results are presented in Table 1.

Setting the gravitational velocity v_G to be equal to the maximum value c, in equation (13), an approximate expression for the parameter ξ for a black hole in oblate spheroidal space time can be obtained as

$$\xi_{blackhole} \approx \frac{2C}{c^2}. \tag{20}$$

Table 1: Basic constants for oblate spheroidal bodies in the solar system

Body	C [$\times 10^{-9}$ Nmkg^{-1}]	D [$\times 10^{-9}$ Nmkg^{-1}]
Sun	−46796.04	−15598.70
Earth	−0.743851	−0.247962
Mars	−0.1132	−0.03780
Jupiter	−3.77107	−1.25803
Saturn	−0.879543	−0.29356
Uranus	−0.842748	−0.28102
Neptune	−1.065429	−0.35516

Table 2: Reduction ratio for oblate spheroidal masses in the solar system to reduce to black holes

Body	$\xi_{surface}$	$\xi_{blackhole}$	reduction ratio: $\frac{\xi_{surface}}{\xi_{blackhole}}$
Sun	241.52	1.1×10^{-3}	2.32250×10^5
Earth	12.01	1.6×10^{-8}	7.50625×10^8
Mars	09.17	2.0×10^{-9}	4.58500×10^9
Jupiter	02.64	8.3×10^{-8}	3.18070×10^7
Saturn	01.97	1.9×10^{-8}	1.03684×10^8
Uranus	03.99	1.8×10^{-8}	2.21667×10^8
Neptune	04.30	2.3×10^{-8}	1.86950×10^8

Hence the parameter $\xi_{blackhole}$ for various bodies in the solar system is computed using equation (20) and the reduction ratio for oblate spheroidal masses in the solar system to reduce to black holes is obtained (Table 2).

The reduction ratio can equally be calculated using Schwarzschild's expression. The equatorial radius (r) for the bodies is divided by the Schwarzschild's radius (r_{schw}) to obtain the reduction ratio. The results are shown in Table 3.

4 Conclusion

This short article presents the notion of black holes in the metric tensor exterior to oblate spheroidal masses. Equation (20) is an approximate expression for the parameter ξ of an oblate spheroid to collapse to a black hole.Reductions ratios computed using the oblate spheroidal metric for Sun and planets in the Solar system authenticates the soundness of metric. The closeness of the reduction ratio for oblate spheroidal masses in the solar system computed using the metric tensor in oblate spheroidal space time to that in Schwarzschild's metric is remarkable.Basically, since the Sun and planets under consideration are oblate spheroidal in nature, the values obtained using the metric tensor contain slight corrections to values obtained using Schwarzschild's metric.

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Table 3: Schwarzschild's reduction ratio

Body	r [$\times 10^3$ m]	r_{schw} [m]	reduction ratio: $\frac{r}{r_{schw}}$
Sun	$700,000$	2.96×10^3	2.36490×10^5
Earth	6378	8.80×10^{-3}	7.24773×10^8
Mars	3396	9.9×10^{-4}	3.43030×10^9
Jupiter	$71,490$	2.8	2.55320×10^7
Saturn	$60,270$	8.5×10^{-1}	7.09059×10^7
Uranus	$25,560$	1.3×10^{-1}	1.96615×10^8
Neptune	$24,760$	1.5×10^{-1}	1.65067×10^8

References

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