Our Mathematical Universe: I. How the Monster Group Dictates All of Physics

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A 4th family b' quark would confirm that our physical Universe *is* mathematical and is discrete at the Planck scale. I explain how the Fischer-Greiss Monster Group dictates the Standard Model of leptons and quarks in discrete 4-D internal symmetry space and, combined with discrete 4-D spacetime, *uniquely* produces the finite group Weyl E_8 x Weyl E_8 = "Weyl" SO(9,1). The Monster's j-invariant function determines mass ratios of the particles in finite binary rotational subgroups of the Standard Model gauge group, dictates Möbius transformations that lead to the conservation laws, and connects interactions to triality, the Leech lattice, and Golay-24 information coding.

1 Introduction

The ultimate idea that our physical Universe *is* mathematical at the fundamental scale has been conjectured for many centuries. In the past, our marginal understanding of the origin of the physical rules of the Universe has been peppered with huge gaps, but today our increased understanding of fundamental particles promises to eliminate most of those gaps to enable us to determine with reasonable certainty whether this conjecture is true or false.

My principal goal is to show that if a 4th quark family exists, the physical rules of the Universe follow directly from mathematical properties dictated by the Fischer-Greiss Monster Group via the Monster's j-invariant function and the Möbius transformation in discrete spacetime, with everything related to the Golay-24 information code for the Leech lattice.

In a series of articles and conference talks beginning in 1992 [1–3] I have been predicting that a 4th quark family with a b' quark at about 80 GeV and a t' quark at about 2600 GeV will be produced at the colliders. Its detection will support these proposals:

1. The Standard Model (SM) of leptons and quarks provides an excellent approximation to the actual *discrete* symmetry groups of these fundamental particles and requires little modification for extension to the Planck scale.
2. There are 3 lepton families and 4 quark families, each family of two states defined by a different finite binary rotational subgroup of the $SU(2)_L$ x $U(1)_Y$ part of the SM gauge group.
3. The leptons are 3-D polyhedral entities, and the quarks are 4-D polytope entities which combine into 3-D colorless hadrons, color being a 4-D property with exact symmetry derived from 4-D rotations.
4. Lepton and quark approximate mass values are determined by the j-invariant function of elliptic modular functions, being related to the above subgroups and Möbius transformations in both discrete lattice spaces and continuous spaces.
5. Both 4-D spacetime and 4-D internal symmetry space are discrete at the Planck scale, and both spaces can be telescoped upwards mathematically by icosians to 8-D spaces that *uniquely* combine into 10-D discrete spacetime with discrete Weyl E_8 x Weyl E_8 symmetry (not the E_8 x E_8 Lie group of superstrings/M-theory).
6. All the above is related to the Fischer-Greiss Monster Group which herein I argue actually dictates all the rules of physics, except perhaps the entropy law.
7. Consequently, our physical Universe *is* mathematical with only one set of rules and physical constants, which eliminates any multiverse with different values.
8. We live in the only possible Universe, the one with 4-D discrete spacetime dictated by the Monster Group and its relation to information coding and the Leech lattice.

My discrete geometrical approach briefly outlined above fits within the realm of the SM, so its past successes should still apply. One simply must "discretize" the SM lagrangian. Even Noether's theorem works in discrete spaces [4] to connect conservation laws to symmetries, the conserved quantity being continuous but periodic.

2 Brief orientation for discreteness

A few years ago a comprehensive review [5] summarized many of the historical mathematical and physical arguments for considering the Universe to be mathematical. Included were the three hypotheses: (1) the External Reality Hypothesis (ERH) — there exists an external physical reality completely independent of us humans; (2) the Mathematical Universe Hypothesis (MUH) — our external physical reality is a mathematical structure; and (3) the Computable Universe Hypothesis (CUH) — the mathematical structure that is our external physical reality is defined by computable functions. Recall that a computable function must be specifiable by a finite number of bits. The mathematical details are in that article.

The ERH is relatively easy to accept, for the universe certainly existed long before we humans came on the scene. The MUH is the conjecture for which I hope the data from the colliders will help us decide. One assumption here is that Gödel's Incompleteness Theorem is not an impediment, i.e., there is no limit to being able to determine the ultimate source of all the rules of Nature and what these rules actually are.

The most interesting statements [5] regarding challenges to the CUH are "... virtually all historically successful theories of physics violate the CUH ..." and "The main source of CUH violation comes from incorporating the continuum, usually in the form of real or complex numbers, which cannot even comprise the input to a finite computation since they generically require infinitely many bits to specify." To me, therein lies the problem: *continuous spaces*.

In particle physics, we consider two spaces: (1) a continuous spacetime for particle movement such as translations, rotations and Lorentz transformations, and (2) a continuous internal symmetry space at each spacetime point for the local gauge interactions of the Standard Model. In both spaces we have successfully used continuous functions for our descriptions of the behavior of Nature.

My proposed solution to this problem is to consider both spaces to be discrete spaces "hidden" underneath the continuous approximation, as if we do not yet have enough resolution to detect this discreteness. All our successful physics theories are then excellent *effective* theories containing continuous fields and continuous wave function amplitudes in this approximate world.

We will not be entering a strange new world by considering a discrete approach, for we use difference equations, lattice models, and discrete computations to approximate continuum physics all the time in numerical calculations, and the results are quite reliable and amazingly accurate. Therefore, I suggest that a fundamental discreteness at the Plank scale of about 10^{-35} meters is not unreasonable [3].

The possibility that the Monster Group, whose influence looms over all of mathematics, could dictate all of physics was put forth in several of my previous papers and conference talks over the last two decades, but other physicists have conjectured a similar proposal. What the others have not realized is the direct connection in a *discrete* internal symmetry space from the Monster to the lepton and quark states via the j-invariant of elliptic modular functions. In this article, I provide additional essential arguments to establish the hegemony of the Monster Group and I arrive at the conclusions that spacetime is discrete and our Universe *is* mathematical.

3 The Monster and the j-invariant

The very large discrete symmetry group called the Monster group M is a finite simple group because it has only two normal subgroups, the trivial one-dimensional group and the whole group itself. Finite simple groups can be used as building blocks in that any other type of finite group can be constructed from them. The list of all finite simple groups is: (i) the cyclic groups C_p, with p prime, (ii) the alternating groups A_n, n > 4, (iii) 16 infinite families of Lie groups, and (iv) 26 sporadic groups. The smallest sporadic is the Mathieu Group M_{11} of order 7920 discovered in 1861, while the largest sporadic is the Monster M constructed in 1980 with order of about 8 x 10^{53}. The Monster has 194 different irreducible representations, with the smallest irreducible matrix representations of M being in space dimensions 1, 196883, 21296876, and 842609326.

As I explain in the next section, the most direct connection of M with the SM of leptons and quarks is via the j-invariant of elliptic modular functions

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots \quad (1)$$

where $q = e^{2i\pi\tau}$ and τ is a ratio for a 2-D lattice that we will define in a later section. I.e., this 2-D lattice approach in our discrete spaces leads directly to the symmetry groups for the lepton and quark families and for the Lorentz transformations in spacetime.

As has been determined by mathematicians, the coefficients of the powers of q are simple linear combinations of dimensions of irreducible representations of the identity operation of M, a correlation known as "Monstrous Moonshine". E.g., 196884 = 1 + 196883, and 21493760 = 21296876 + 196883 + 1, etc. More mathematical and historical information about the Monster can be learned from the online papers and books by T. Gannon [6].

4 Binary rotation groups and the j-invariant

Here I review the connection between the j-invariant and the discrete symmetry groups for the leptons and quarks. I have proposed [1–3] that the lepton and quark flavors, being electroweak eigenstates, correspond to orthogonal states in specific discrete symmetry groups called finite binary rotational groups. These seven subgroups of the SM local gauge group act in the $\mathbb{R}^3$ and $\mathbb{R}^4$ real subspaces of the 2-D unitary space $\mathbb{C}^2$ for $SU(2)_L$ x $U(1)_Y$. In fact, I am using discrete $\mathbb{R}^3$ and $\mathbb{R}^4$.

The lepton families correspond to the 3-D finite binary rotational groups called the binary tetrahedral group 2T, the binary octahedral group 2O, and the binary icosahedral group 2I, also labelled as [3, 3, 2], [4, 3, 2], and [5, 3, 2], respectively, in Table 1. These are groups of discrete symmetry rotations and reflections. Binary here refers to the double cover of the SO(3) rotation group by Spin(3) = SU(2), so these groups are finite subgroups of SU(2) and $SU(2)_L$ x $U(1)_Y$.

Having exhausted the group possibilities in $\mathbb{R}^3$, one moves up one real spatial dimension to $\mathbb{R}^4$ in order to define the quark families, which then correspond to the finite binary rotation groups [3, 3, 3], [4, 3, 3], [3, 4, 3], and [5, 3, 3] of the regular 4-D convex polytopes. One may not need the number of quark families to match the number of lepton families for anomaly cancellation because this geometrical approach defines leptons and quarks as 3-D and 4-D entities, respectively. I.e., the interactions are not among point particles.

Leptons						Quarks					
group	order	family	N	Pred. Mass (MeV)	Emp. Mass (MeV)	**group**	order	family	N	Pred. Mass (GeV)	Emp. Mass (GeV)
					.						.
					.	**[3, 3, 3]**	120	$d^{-1/3}$	1/4	0.011	0.007
					.			$u^{+2/3}$		0.38	0.004
[3, 3, 2]	24	e^-	1	[1]	0.511	**[4, 3, 3]**	384	$s^{-1/3}$	1	0.046	0.2
		ν_e		0?	0.0?			$c^{+2/3}$		[1.5]	1.5
[4, 3, 2]	48	μ^-	108	108	103.5	**[3, 4, 3]**	1152	$b^{-1/3}$	108	[5]	5.0
		ν_μ		0?	0.0?			$t^{+2/3}$		160	171.4
[5, 3, 2]	120	τ^-	1728	1728	1771.0	**[5, 3, 3]**	14400	$b'^{-1/3}$	1728	~ 80	?.?
		ν_τ		0?	0.0?			$t'^{+2/3}$		~ 2600	?.?

Table 1: Lepton and quark families for the binary rotational groups [a, b, c], their j-invariant proportionality constant N, and the predicted mass values for the quarks based upon group-to-group N ratios with the charm quark mass [1.5 GeV] and bottom quark mass [5 GeV] as reference masses for ratios of the "up-like" and "down-like" quark states, respectively. These are the "bare" mass predictions. Drawings with these symmetries are online [3].

Each lepton group represents the *binary* rotational symmetries of familiar 3-D regular polyhedrons, the tetrahedron, the octahedron, and the icosahedron. In terms of two complex variables z_1 and z_2, there are three algebraic equations for each regular polyhedron that remain invariant under the operations of its binary group, corresponding to the complex equations for the vertices, the face centers, and the edge centers. Call these three equations W_1, W_2 and W_3, respectively. F. Klein, in a famous 1884 book [7], reported that these three equations are not independent because they form a mathematical syzygy. He showed that two independent equations W_1 and W_2, say, have a ratio proportional to the j-invariant

$$j(\tau) = \frac{W_1}{NW_2} \qquad (2)$$

where N is a specific integer, being 1, 108, and 1728, for the three groups, 2T, 2O, and 2I, respectively. Certain integrals, including a mass integral, for the particle states would involve these N values as important factors.

The four binary rotational groups for the quarks are handled [8] by projecting their physical 4-D polytopes onto the 2-D unitary plane $\mathbb{C}^2$ and realizing that their symmetries lead to the same invariant algebraic equations as for the leptons, with the addition of one other symmetry group syzygy for [3, 3, 3]. The corresponding N values are thus 1/4, 1, 108, 1728.

These N values suggest the pairings of the lepton families to quark families as shown horizontally in Table 1. Notice that these family pairings are different from the traditional ad hoc pairings that are normally suggested for the SM because here there exist fundamental geometrical connections.

5 Particle mass values

The influence of the j-invariant of the Monster continues. In spaces where the j-invariant applies, all rational functions (ratio of two polynomials) are proportional to the j-invariant and invariant under all fractional linear transformations (also called Möbius transformations). For physics purposes, mass of a fundamental particle is proportional to the j-invariant because mass is an invariant under Möbius transformations. Conservation laws in physics can be related to Möbius transformations in both discrete and continuous spaces.

At this stage there is no absolute mass scale, so I must use mass ratios only, selecting a different reference mass value for the "up" states and for the "down" states. For the lepton mass values, we have the N ratios 1:108:1728. Table 1 shows the predicted and the actual values. The patterns of ratios match roughly and they were the clue to considering these binary rotational groups.

Note that without using the reference empirical masses for the ratios, the two predicted states in each family would be degenerate with the same mass. One should form two new orthogonal linear superposition states from these original degenerate states. These states would have different "bare mass" values and would be sensitive to the "vacuum" environment.

For the electroweak interactions, a zero-order approximation to the quark CKM mixing matrix and the lepton PMNS mixing matrix follows directly from the characteristic equations of the 3-D and 4-D symmetries projected to the unitary plane $\mathbb{C}^2$, producing unitary eigenvectors and eigenvalues λ_j = exp[iϵ_j]. The two angles (ϵ_1, ϵ_2) are (π, π) for [3,3,2], (2π/3,

$4\pi/3$) for [4,3,2], ($2\pi/5$, $8\pi/5$) for [5,3,2], ($2\pi/5$, $8\pi/5$) for [3,3,3], ($\pi/3$, π) for [4,3,3], ($\pi/6$, $7\pi/6$) for [3,4,3], and ($\pi/15$, $19\pi/15$) for [5,3,3].

One can define a 3 x 3 unitary matrix [9] and substitute angle difference values for the lepton mixing matrix PMNS and a 3 x 3 quark mixing matrix CKM3, producing

$$PMNS = \begin{pmatrix} 0.5 & 0.866 & \epsilon \\ -0.579 & 0.335 & 0.743 \\ 0.643 & -0.372 & 0.669 \end{pmatrix} \tag{3}$$

$$CKM3 = \begin{pmatrix} 0.978 & 0.208 & \epsilon \\ -0.180 & 0.847 & 0.5 \\ 0.104 & -0.489 & 0.866 \end{pmatrix} \tag{4}$$

with ϵ small. Several of the off-diagonal values in VCKM3 would require higher order corrections in order to better agree with empirically determined values.

A 4 x 4 unitary mixing matrix for our four quark families that brings in c_{34} and s_{34} in the 3rd and 4th rows leads to

$$VCKM4 = \begin{pmatrix} 0.978 & 0.208 & \epsilon_1 & \epsilon_2 \\ -0.180 & 0.847 & 0.5 & \epsilon_3 \\ 0.099 & -0.465 & 0.842 & 0.309 \\ -0.032 & 0.151 & -0.268 & 0.951 \end{pmatrix} \tag{5}$$

with all the ϵ values small. Adjustments can be made by considering higher order corrections.

One should not ignore the fact that a degrees-of-freedom argument would make neutrinos that are zero mass exactly. My two lepton states in each family have 4 d.o.f. total, which can partition into the massive electron state with 3 d.o.f., leaving just 1 d.o.f. for the neutrino state. Thus, the neutrino is massless and can have one helicity state only. Alternately, if both lepton states per family share the 4 d.o.f. equally with 2 d.o.f. each, then these would be two massless states, i.e., possibly two sterile neutrino states. Nature appears to have chosen the unequal split, but sterile neutrinos are still a possibility. As to the quarks, the two 4-D quark family states have a total of 6 d.o.f. to split 3-3, guaranteeing the existence of the two massive quark states per family we measure.

The discovery of the b' quark, probably by the FCNC decay b' $\to$ b + γ, is the acid test of this geometrical approach toward understanding the SM. There is already some hint in the Fermilab data for this decay but the signal/noise ratio is not good enough. The 4th quark family has recently been in vogue because the baryonic particle-antiparticle asymmetry in the Universe (BAU) can then be explained by CP violation with a new value for the Jarlskog invariant that is about 10^{13} times larger [10] than for only 3 quark families. As far as I know, the b' quark remains a viable possibility.

6 Discrete internal symmetry space

In this geometrical approach, the internal symmetry space is discrete $\mathbb{C}^2$ at the Planck scale. Therefore we must consider the mathematical properties of a 2-D hexagonal lattice (or of a 2-D rectangular lattice) of mathematical nodes either with two real axes $\mathbb{R}^2$, or two complex axes $\mathbb{C}^2$, or two quaternion axes $\mathbb{H}^2$, etc. All its nodes can be represented by integer linear combinations of two complex numbers that we label ω_1 and ω_2 forming a right-handed basis (ω_1, ω_2). We can change these two numbers without changing the lattice by letting

$$\begin{aligned} \omega_1' &= a\omega_1 + b\omega_2 \\ \omega_2' &= c\omega_1 + d\omega_2 \end{aligned} \tag{6}$$

where a, b, c, and d, are integer elements of a 2 x 2 matrix with determinant 1. Such matrices form a symmetry group called the "modular group" SL(2, Z) which is related to elliptic curves. Actually, all that matters is the ratio $\tau = \omega_1/\omega_2$ which defines the τ for the j-invariant in Eq. 1. Since

$$f(\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right), \tag{7}$$

all modular functions $f(\tau)$ on the lattice depend only upon its shape. The j-invariant is such a function, and all other SL(2, Z)-invariant functions are rational functions of $j(\tau)$.

Eq. 7 defines the fractional linear transformations, i.e., the Möbius transformations, which are based upon the transformations $\tau \to 1 + \tau$ and $\tau \to -1/\tau$ for translations, rotations, etc. In the limit when the node spacing approaches zero, the continuous approximation appears and the Möbius transformations include the continuous symmetry transformations.

7 Geometry of the boson interactions

The 12 bosons of the SM, 8 gluons and 4 EW bosons, operate on the fermion states in a continuous internal symmetry space. For a continuous space one can map the complex plane $\mathbb{C} = \mathbb{R}^2$ and unitary plane $\mathbb{C}^2 = \mathbb{R}^4$ to the 2-D Riemann sphere. Its 2-D surface has no demarcations, thus allowing any small or large rotation. Consequently, the symmetry group for the SM interactions is the continuous gauge group of operations.

In my geometrical approach this internal symmetry space is discrete, so only specific finite rotation groups can produce these boson operations. However, when the internal symmetry space is discrete and particle symmetries are defined by the specific finite binary rotation groups for leptons and quarks, the Riemann surface is tessellated, i.e., composed of identical equilateral triangles, their number uniquely determined by the binary rotation group. Then the number of rotational operations becomes severely restricted and each boson operator must respect the integrity of the symmetry group for the lepton or quark families participating in the interaction.

Geometry provides the important clue. We desire a small group in our discrete space for defining these interactions (i.e., producing the appropriate rotations by the bosons), and we find the binary icosahedral group 2I or [5, 3, 2]. However, there will be some missed operations on the symmetry for the binary octahedral group 2O. But if we take 2I twice, i.e., including its "reciprocal" [5, 3, 2], then we get it all.

In order to appreciate this geometry, quaternion algebra simplifies the game. Recall that the SU(2) matrix representation and the unit quaternion **q** are related by

$$\mathbf{q} = w\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Longleftrightarrow \begin{pmatrix} w + iz & x + iy \\ -x + iy & w - iz \end{pmatrix} \quad (8)$$

where the **i**, **j**, and **k** are unit imaginaries, the coefficients w, x, y, and z are real, and $w^2+x^2+y^2+z^2 = 1$. We can represent the two orthogonal lepton or quark states in each family by two orthogonal unit quaternions in $\mathbb{C}^2$.

There is also a conjugate plane $\mathbb{C}'^2$ for the antiparticles and its Riemann sphere. The conjugate quaternion is **q'** =w**1** - x**i** - y**j** - z**k**. What we discuss for the particle states works for the antiparticle states, too. Having a conjugate space is very special. Clifford algebra and Bott periodicity dictate that only $\mathbb{R}^4$, $\mathbb{R}^8$, and other real spaces $\mathbb{R}^n$ with dimensions divisible by four have two equivalent conjugate spaces. This specific mathematical property dictates a world with both particle states and their antiparticle states for these dimensions only.

One more mathematical fact. The group $U(1)_Y$ for weak hypercharge Y in $SU(2)_L$ x $U(1)_Y$ has the important role of reducing the symmetry between the two spaces, normal and conjugate, in $\mathbb{R}^4 = \mathbb{C}^2$ from being simply equivalent to their being *gauge* equivalent. The physics consequence is that particles and antiparticles have the same positive mass but all other properties can be opposite sign. Alternately, we can use the 2-element inversion group C_i to accomplish the same distinction as well as to determine the intrinsic parity of the particle states, odd for particles and even for antiparticles.

Furthermore, the use of quaternions for the electroweak operations tells us that the L in $SU(2)_L$, which means left-handed chirality only for the weak interaction, is really dictated by quaternion properties, so that the left-handed physics restriction for the weak interaction in $\mathbb{C}^2$ follows. That is, in the normal unitary plane all unit quaternions have left-handed screw transformations that mix the two orthogonal states and right-handed screw transformations that do not. Put another way, the quaternions transform the two orthogonal flavor states as left-handed doublets and right-handed singlets. For example, in the first lepton family, they are (ν_{eL}, e_L) and (ν_{eR}) and (e_R). In the conjugate unitary plane for antiparticles, the quaternion transformations have the opposite handedness.

Now back to rotating the Riemann sphere. In the simplest electroweak (EW) interactions of a boson with an incoming fermion, the fermion state either remains the same (via γ or Z^0) or changes from the initial state to an orthogonal state (via $W^\pm$). As examples, the γ may be the identity and the Z^0 may produce a 4π rotation, while the $W^\pm$ operates between different states. The 120 operations of the binary icosahedral group 2I are represented by 120 unit quaternions, and 2I contains almost all the rotation operations needed for the 7 fermion family groups. However, several symmetry operations of 2O would be absent. One needs to add the "reciprocal" binary icosahedral group to include all the operations of 2O, making a grand total of 240 operations. (n.b. One could also consider just the generators to realize the same result.)

Here comes an interesting and unexpected mathematical consequence. The first set of 120 quaternions can be expressed as 120 special unit quaternions known as *icosians* which telescope 4-D discrete-space quaternions up to being 8-D discrete-space octonions to locate points that form a special lattice in $\mathbb{R}^8$ called D_8. The second set of 120 quaternions does the same, forming another D_8 lattice in $\mathbb{R}^8$ by filling the holes in the first D_8 lattice.

The icosians are special unit quaternions q_i which have the mathematical form

$$\begin{aligned} q_i = (e_1 + e_2 \sqrt{5}) + (e_3 + e_4 \sqrt{5})i + \\ +(e_5 + e_6 \sqrt{5})j + (e_7 + e_8 \sqrt{5})k \end{aligned} \quad (9)$$

where the eight e_j are special rational numbers. The important mathematical fact here is that in each pair, such as (e_3 + $e_4 \sqrt{5}$), exactly one of the e_j is nonzero. Therefore, even though the icosians are telescoping us up to an 8-D space, their primary importance is that they represent 4-D operations in $\mathbb{R}^4$ even though we can now define identical quaternion operations via octonions in the much larger $\mathbb{R}^8$ space also.

Together, these two D_8 lattices of 120 icosians each combine to form the 240 octonions that define the famous E_8 lattice in $\mathbb{R}^8$. The symmetry group for this E_8 lattice is not the Lie group E_8 but the discrete group Weyl E_8.

Therefore, the operations of the SM occur in discrete 4-D internal symmetry space, but they operate also in the discrete 8-D space because these icosians span both spaces simultaneously.

8 Quark color, gluons, and hadron states

Now I must back up to show that the gluon interactions can occur in $\mathbb{R}^4$ for $SU(3)_C$ even though one normally expects the larger space $\mathbb{C}^3$. Because 4-D rotations are simultaneous rotations in two orthogonal planes, each of the three quark color charges Red, Green, and Blue, can be assigned to the three possible rotation plane pairs [wx, yz], [xy, zw], and [yw, xz], respectively. Actually, because these three 4-D rotation pairs are equivalent and we could have made the color assignments in any order, we learn the mathematical reason for color being an *exact* physical symmetry.

Contained within the above specific icosians are the gluon operations on the color states, but one can use a specific 4 x 4 rotation block matrix R to define the transition from one color state in the 4-D space to another. There are 8 orthogonal gluon matrices in agreement with the 8 gluons of the $SU(3)_C$ gauge group of the SM.

Hadrons are colorless quark combinations, so they occur when the combined resultant 4 x 4 matrices produce no net 4-D rotation, i.e., are the identity matrix. One can show that this colorless state exists for three combinations of quark states only: (1) the quark-antiquark pair with color and anticolor, (2) three quarks, or (3) three antiquarks, with the appropriate linear combinations of colors or anticolors.

The mathematics itself distinguishes quarks (and baryon number) from leptons: the quarks are 4-D entities and the leptons are 3-D entities, with only the 4-D entities capable of the color interaction because color is an exact symmetry in $\mathbb{R}^4$. Quark *confinement* results because isolated quarks are 4-D entities which cannot exist in a 3-D space, so one can never have an isolated single quark in our 3-D spatial world.

The colorless hadron states, being those special mathematical combinations of quark 4-D entities, are now actually 3-D entities like the lepton states are. That is, the colorless combinations of quarks are 3-D composite particle states because their geometrical intersections define 3-D geometric entities.

Therefore, in my geometrical version of the SM, we have 3-D lepton states, 3-D hadron states, 3-D electroweak boson states, but 4-D quark states and 4-D gluon states. The 4-D quark and gluon states are confined, i.e., they cannot exist as separate entities in our 3-D spatial world, but the 3-D lepton, 3-D hadron, and 3-D electroweak boson states can move through 4-D discrete spacetime with its 3 spatial dimensions.

9 Geometry of discrete 4-D spacetime

Our 3-D particles move in discrete 4-D spacetime. We know that continuous 4-D spacetime has symmetries related to its continuous Lorentz group SO(3,1). For a discrete 4-D spacetime and its Lorentz transformations we need to determine a finite subgroup of SO(3,1) for its discrete symmetry.

A clever mathematical approach to 4-D spacetime was introduced by R. Penrose [11] long ago, who showed how to utilize his “heavenly sphere” to account for Lorentz transformations, etc. This “heavenly sphere” is actually 4-D spacetime (t, x, y, z) mapped onto the Riemann sphere. Consider being in the center of the “heavenly sphere” so that light rays from stars overhead pass through unique points on the unit celestial sphere surrounding you. A Lorentz boost is a conformal transformation of the star locations: the constellations will look distorted because the apparent lengths of the lines connecting the stars will change but the angles between these connecting lines will remain the same.

In our discrete 4-D spacetime we need to tesselate this Riemann surface into identical equilateral triangles and then perform the symmetry transformations of the sphere. But we have already achieved this tesselation earlier with the binary rotation groups when we considered the discrete internal symmetry space mapped to the Riemann sphere, so we know the result. Using the isomorphism SO(3,1) = PSL(2,C), we see [2] that the group mathematics connects the conformal transformations just described to the Möbius group via

$$SO(3,1) = \text{Möbius group} = PSL(2,\mathbb{C}), \qquad (10)$$

with the discrete Lorentz transformations of the tessellated Riemann sphere already contained in SO(3,1). Thus, we have a unit quaternion group PSL(2,C) (equivalently, an SU(2) matrix or spinor) representation of the Lorentz transformation.

Therefore, we are back to our discrete symmetries of the binary polyhedral groups because they are finite modular subgroups of the Möbius group PSL(2,C). Therefore, the 240 special quaternions called icosians are now required for discrete Lorentz boosts and discrete rotations in the discrete 4-D spacetime. We obtain a second E_8 lattice in $\mathbb{R}^8$ with symmetry group Weyl E_8.

10 Unification of spacetime and the Standard Model

We can now unite the discrete internal symmetry space operations with the discrete spacetime operations [2]. The direct product of our two Weyl E_8 groups results in a subgroup of the continuous group PSL(2,$\mathbb{O}$), where $\mathbb{O}$ represents all the unit octonions. For the continuous case, PSL(2,$\mathbb{C}$) has become PSL(2,$\mathbb{O}$) = SO(9,1), the Lorentz group in 10-D spacetime. That is, the final combined spacetime is *bigger* than I expected, being isomorphic to a 10-D spacetime instead of an 8-D spacetime.

Applying this result to our discrete case, the combined finite subgroup

$$\text{finite } PSL(2,\mathbb{O}) = \text{finite } SO(9,1), \qquad (11)$$

the finite Lorentz group in discrete 10-D spacetime. The same result, expressed in terms of the direct product of the Weyl E_8 groups is

$$\text{Weyl } E_8 \text{ x Weyl } E_8 = \text{“Weyl''} SO(9,1), \qquad (12)$$

a finite subgroup of SO(9,1).

Therefore, the big surprise is that the combination of a 4-D discrete spacetime with a 4-D discrete internal symmetry space creates a *unique* connection to 10-D discrete spacetime, not to an 8-D discrete spacetime. Unlike the situation with continuous spaces, we do not have a 6-D “curled up” internal symmetry space with about 10^{500} possibilities.

The mathematics has dictated a beautiful result: there seems to be *only one way* for our Universe to exist when spacetime is discrete.

11 A physical particle model

Even though the mathematics telescopes us up from $\mathbb{R}^4$ to $\mathbb{R}^8$, we still need a physical model of particles in the discrete 4-D spacetime defining our Universe. The leptons, hadrons, and the electroweak bosons are non-point-like 3-D entities that appear to be point-like particles at our normal size resolution of about 10^{11} times larger than the Planck scale.

Peering in at the Planck scale, however, I expect the discrete 4-D internal symmetry space at each spacetime point to conjoin into the discrete 4-D spacetime. In order to do so, each particle must *emerge* by "gathering in" nodes of the lattice to make its 3-D or 4-D entity with its correct symmetry. For example, if the particle is an electron, we expect the symmetry of the node collection will be tetrahedral to agree with its [3, 3, 2] symmetry. If the lattice of nodes was originally uniformly spaced in this region of discrete space, then the existence of the electron has distorted this lattice with a decreasing distortion amount for increasing distance from the electron's center.

Note that this geometrical approach assumes that the lattice nodes themselves do not have any *measurable* physical properties. Consequently, we have arrived finally at the end of the hierarchy of physical particles within particles. At this point in the geometrical approach we simply must accept this gathering-of-nodes process because the mathematics dictates this process via graph theory and Kuratowski's theorem.

Kuratowski's theorem is important here because it states that a graph is planar if and only if it does not contain a Kuratowski subgraph K_5 or $K_{3,3}$. For example, if an n-dimensional graph (a lattice of nodes) in a spatial dimension higher that 2-D does not contain a Kuratowski K_5 subgraph, also known as the complete graph of five vertices, then this n-D graph reduces to 2-D.

But the first quark family's binary rotational group [3, 3, 3] symmetry is the rotationally symmetric version of the Kuratowski subgraph K_5. Therefore, at least one quark state of the first quark family is stable as it moves through the lattice, while all other quark families have states that will decay down to [3, 3, 3] quark states. Indeed, the physics agrees with this mathematical prediction.

At the DISCRETE'08 conference in December, 2008 where I tried to present this geometrical approach in my allotted 20 minutes (!), C. Jarlskog asked me an interesting question: Why can't the universe have only quarks and gluons? I.e., a QCD world seems complete by itself. Why complicate the material world with leptons and the electroweak interaction? To which I immediately answered: Kuratowski's Theorem in mathematics does not allow such a world, but I was not encouraged to elaborate with any of the details.

Here is the rest of my argument. If quarks are 4-D entities, most quark states decay because they do not have the structure of K_5 (or $K_{3,3}$), so the initial structure will re-form into two or more new particles. In a universe with only quarks and gluons, a problem arises because gluons change only the color state for a particular quark but cannot change one quark flavor into another. In order to obey Kuratowski's theorem, Nature had no choice but to bring in more particles, notably the leptons and the electroweak interaction bosons. Voilà!

The immortality of the electron with group [3,3,2] seems to depend upon its close geometrical relation to the regular K_5 symmetry group [3, 3, 3]. Of course, the electron could annihilate with its antiparticle (and so can a quark).

At this point one might be concerned about the emergence of fermion particles from the "vacuum" state. In order to account for all the particles in the known Universe, the equivalent of about one new hydrogen atom per cubic meter per 10 million years is required. This process can occur because fermions are represented by spinors, and spinors originate from zero-length vectors. That is, according to E. Cartan, one zero-length vector splits into a spinor and conjugate spinor mathematically. The spinor is the fermion such as an electron and the conjugate spinor is the anti-fermion positron, for example. If their total energy remains zero by adding up all energy forms, then this creation process is viable.

As the electron or any 3-D particle moves through the lattice, I would expect that the particle's lattice distortion effect moves with it, with its previous distortions relaxing back toward being a regular lattice while the oncoming positions become more distorted. Mathematically, the Möbius transformations guarantee the integrity of this movement. That is, for our lattice, the transformation $\tau \to 1+\tau$ ensures that the movement process is identical everywhere in the lattice. The second Möbius transformation $\tau \to -1/\tau$ when combined with the one above allows rotations and other linear transformations to occur in the lattice.

This lattice distortion by a particle in 4-D discrete spacetime is the "warping of spacetime" associated with the gravitational interaction proposed by A. Einstein in the general theory of relativity. In this way, gravitation appears to be different from the other fundamental interactions which appear to be more localized.

More details of this particle model, such as the geometry of the gravitational interaction, the origin of the rules of quantum mechanics, the origin of time, and the information coding of the fundamental particles, will be discussed thoroughly in the second paper of this series.

12 Triality, the Leech lattice, and information coding

We know that particle EW interactions can be described in lowest order by the Feynman diagram (Fig. 1) involving three particles with three lines meeting at a point. There can be two fermions interacting with one of the electroweak ($W^{\pm}$, Z^0, or γ) or color (8 gluons) bosons. There can be three gluons interacting. More complicated diagrams can be drawn but they will all be made from combinations of this generic one.

This lowest order Feynman diagram with two fermions and one boson is a mathematical *triality* diagram with the fermions representing spinors and the boson representing a vector Jordan algebra entity. Triality is a relationship between three vector spaces over a field F that are all isomorphic to each other. Thus, the common vector space is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, i.e., involving spinors in dimensions 1, 2, 4, and 8, respectively [12].

In our 4-D discrete spacetime the fermion states can be represented by quaternions. In fact, Clifford algebra tells us that there will be two quaternion representations in $\mathbb{R}^4$ called the right-handed spinor representation S_4^+ and the left-handed

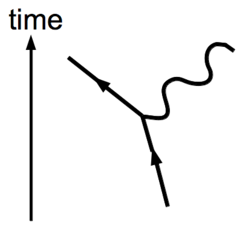

Fig. 1: The incoming fermion emits or absorbs a boson and a fermion exits. E.g., an electron emits a photon and continues in a different direction. This diagram represents triality among two spinors (electron in and electron out) and a vector boson.

spinor representation S_4^-. In general, for even dimensional spacetimes, i.e., even n values, the two spinor representations have dimension $2^{n/2-1}$, but the vector representation has dimension n. For example, in n = 4 space, the boson vector representation is a 4 x 4 real matrix and the fermion spinor representation is a 2 x 2 complex matrix or, equivalently, also a 4 x 4 real matrix. I.e., the fermions and bosons are the same dimension.

We know that the icosians telescope us up to discrete 8-D. With n = 8, the spinor representations are again the same size as the vector representation, both represented by 8 x 8 real matrices. Even so, they are not equivalent representations. However, one can permute the vector, left-handed spinor, and right-handed spinor representations into each other [12]. In 4-D, for example, there is a parity operator that can do this change of a left-handed spinor into a right-handed spinor and vice-versa.

For the generic Feynman diagram, one can think about the two fermions and the one boson as being three E_8 lattices which come together momentarily to form a 24-D lattice called the Leech lattice. The Monster Group again plays its governing role through the j-invariant function. The numerator of $j(\tau)$, being $1 + 720\ q + 146512\ q^2 + \ldots$, is the generating function for the lattice vectors in this product of three copies of the E_8 lattice. And for conformal field theories, the j-invariant is the partition function for the Monster Group [13].

Another very important mathematical connection takes us to information coding theory. One could say that each particle in the triality diagram brings in its 8-bit Hamming code word to temporarily form the 24-bit binary Golay code word or, equivalently, the 12-bit ternary Golay code word, related to the Leech lattice. The 8-bit Hamming code has 72 distinct code words in 9 different but overlapping sets [14], the exact number required for the fundamental particles of the SM: 6 leptons plus 8 x 3 = 24 quarks sums to 30 fermion states; when doubled for anti-particles, makes 60 particle states; then add the 12 bosons to get 72. The 24-bit Golay code word encodes 12 data bits defining up to $2^{12} = 4096$ different items, easily covering the possible interaction triples of the SM.

These code words support the hegemony of the Monster Group because the allowed SM interactions of the leptons and quarks can be related to information theory in 24 dimensions. The second article includes details of the Turyn construction for these Golay-Leech lattice code words and their relationship to quantum information theory and the Monster Group.

13 Conclusion

In this brief article I have outlined specific connections between the mathematics of the Monster Group and fundamental physics particles and rules. These connections support the three hypotheses ERH, MUH, and CUH, so I conclude that the Universe *is* mathematical and that we live in the only possible one. I await the empirical confirmation by the discovery of the 4th quark family, particularly the b' quark at about 80 GeV. Hopefully, the wait will not be long.

Acknowledgements

The author acknowledges Sciencegems.com grant support.

Submitted on June 26, 2011 / Accepted on July 03, 2011

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