Relativistic Dynamical Theory for Test Particles and Photons in Static Spherically Symmetric Gravitational Fields

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The gravitational line element in this field is used to postulate the four spacetime element of arc vector, volume element, del operator and divergence operator for space-time gravitational fields. A relativistic dynamical theory is then established for static spherically symmetric gravitational fields. Equations of motion for test particles and photons are obtained with post Newton and post Einstein correction terms of all orders of c^{-2}.

1 Introduction

Schwarzschild in 1916 constructed the first exact solution of Einstein's gravitational field equations. It was the metric due to a static spherically symmetric body situated in empty space such as the Sun or a star [1].

In this article, we establish a link between Schwarzschild's metric and Newton's dynamical theory of gravitation. The consequence of this approach is the emergence of complete expressions for the velocity, acceleration and total energy with post Newton and post Einstein correction terms to all orders of c^{-2} [2].

2 Euclidean Geometry in Static Spherically Symmetric Fields

Recall that the scalar world line element dS^2 in Schwarzschild's gravitational field is given as

$$dS^2 = -g_{11}dr^2 - g_{22}d\theta^2 - g_{33}d\phi^2 + g_{00}(dx^0)^2 \qquad (2.1)$$

where

$$g_{00} = \left(1 - \frac{2GM}{c^2 r}\right),$$

$$g_{11} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1},$$

$$g_{22} = r^2,$$

$$g_{33} = r^2 \sin^2\theta.$$

G is the universal gravitational constant, c is the speed of light in vacuum and M is the mass of the static homogeneous spherical mass (Schwarzschild's mass) [3, 4]. Now, also recall that the world line element dS^2 from which the metric tensor is formulated is obtained from the fundamental line element $d\overline{S}(r,\theta,\phi)$. Also, from vector analysis, it is well known that $d\overline{S}(r,\theta,\phi)$ is the most fundamental quantity from which all vector and scalar quantities required for the formulation of the dynamical theory of classical mechanics are derived.

2.1 Element of arc vector

From equation (2.1), we realise that Schwarzschild's gravitational field is a four dimensional orthogonal vector space with coordinates (r,θ,ϕ,x^0) and unit vectors $(\widehat{r},\widehat{\theta},\widehat{\phi},\widehat{x^0})$ and hence the element of arc vector $d\overline{S}$ is given as

$$d\overline{S} = [-g_{11}]^{1/2}(dr)\widehat{r} + [-g_{22}]^{1/2}(d\theta)\widehat{\theta} + [-g_{33}]^{1/2}(d\phi)\widehat{\phi} + [g_{00}]^{1/2}(dx^0)\widehat{x^0} \qquad (2.2)$$

with scale factors h_r, h_θ, h_ϕ and h_{x^0} defined as

$$h_r = [-g_{11}]^{1/2},$$

$$h_\theta = [-g_{22}]^{1/2},$$

$$h_\phi = [-g_{33}]^{1/2},$$

$$h_{x^0} = [g_{00}]^{1/2}.$$

2.2 Volume element and Gradient operators

As in Eulidean geometry in three dimensional vector space, we postulate that the volume element dV in Schwarzschild's gravitational field is given by

$$dV = dS_r dS_\theta dS_\phi dS_{x^0} \qquad (2.3)$$

and the corresponding space element of volume

$$dV = dS_r dS_\theta dS_\phi, \qquad (2.4)$$

where

$$dS_r = h_r dr,$$

$$dS_\theta = h_\theta d\theta,$$

$$dS_\phi = h_\phi d\phi,$$

$$dS_{x^0} = h_{x^0} dx^0.$$

We postulate that our complete spacetime del operator in Schwarzschild's gravitational field is given as

$$\overline{\nabla} = \frac{\widehat{r}}{h_r}\frac{\partial}{\partial r} + \frac{\widehat{\theta}}{h_\theta}\frac{\partial}{\partial\theta} + \frac{\widehat{\phi}}{h_\phi}\frac{\partial}{\partial\phi} + \frac{\widehat{x^0}}{h_{x^0}}\frac{\partial}{\partial x^0}. \qquad (2.5)$$

The complete spacetime divergence, curl and laplacian operators can be defined in a similar manner[2].

3 Relativistic Dynamical Theory for Test Particles

From the spacetime line element, the instantaneous spacetime velocity vector in the gravitational field can be defined[2] as

$$\bar{u} = \frac{d\bar{S}}{d\tau} \tag{3.1}$$

or

$$\bar{u} = u_r\hat{r} + u_\theta\hat{\theta} + u_\phi\hat{\phi} + u_{x^0}\hat{x}^0, \tag{3.2}$$

where τ is the proper time,

$$u_r = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \dot{r},$$

$$u_\theta = r\dot{\theta},$$

$$u_\phi = r\sin\theta\dot{\phi}$$

and

$$u_{x^0} = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \dot{x}^0.$$

Hence, the instantaneous speed u is

$$u^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 + \left(1 - \frac{2GM}{c^2 r}\right)(\dot{x}^0)^2. \tag{3.3}$$

Also the instantaneous spacetime acceleration vector is given as

$$\bar{a} = \frac{d\bar{u}}{d\tau} \tag{3.4}$$

or

$$\bar{a} = a_r\hat{r} + a_\theta\hat{\theta} + a_\phi\hat{\phi} + a_{x^0}\hat{x}^0, \tag{3.5}$$

where

$$a_r = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \ddot{r} - \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-3/2} \dot{r}^2,$$

$$a_\theta = r\ddot{\theta} + \dot{r}\dot{\theta},$$

$$a_\phi = \dot{r}\sin\theta\dot{\phi} + r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi}$$

and

$$a_{x^0} = \frac{d}{d\tau}\left[\left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \dot{x}^0\right].$$

Now, recall that the inertial mass m_I and passive mass m_p are related to the rest mass m_0 of a particle by

$$m_I = m_p = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} m_0 \tag{3.6}$$

where in this gravitational field, u^2 is as defined in equation (3.3). Also, the linear momentum of a particle of nonzero rest mass is defined as

$$\bar{P} = m_I\bar{u} \tag{3.7}$$

or

$$\bar{P} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} m_0\bar{u}. \tag{3.8}$$

The instantaneous relativistic kinetic energy (T) of a particle of nonzero rest mass is given as

$$T = (m_I - m_0)\,c^2 \tag{3.9}$$

or

$$T = \left[\left(1 - \frac{u^2}{c^2}\right)^{-1/2} - 1\right] m_0 c^2 \tag{3.10}$$

and the instantaneous relativistic gravitational potential energy (V_g) for a particle of nonzero rest mass is

$$V_g = m_p\Phi = -\left(1 - \frac{u^2}{c^2}\right)^{-1/2} \frac{GMm_0}{r}, \tag{3.11}$$

where $\Phi = \frac{-GM}{r}$ is the gravitational scalar potential in Schwarzschild's gravitational field. Thus, the total relativistic mechanical energy E for a particle of nonzero rest mass is given as

$$E = T + V_g \tag{3.12}$$

or

$$E = m_0 c^2\left[\left(1 - \frac{GM}{c^2 r}\right)\left(1 - \frac{u^2}{c^2}\right)^{-1/2} - 1\right]. \tag{3.13}$$

Thus, our expression for total energy has post Newton and post Einstein correction terms of all orders of c^{-2}.

The relativistic dynamical equation of motion for particles of non-zero rest mass[2] is given as

$$\frac{d}{d\tau}\bar{P} = -m_p\bar{\nabla}\Phi \tag{3.14}$$

or

$$\frac{d}{d\tau}\left[\left(1 - \frac{u^2}{c^2}\right)^{-1/2} m_0\bar{u}\right] = -\left(1 - \frac{u^2}{c^2}\right)^{-1/2} m_0\bar{\nabla}\Phi \tag{3.15}$$

or

$$\bar{a} + \frac{1}{2c^2}\left(1 - \frac{u^2}{c^2}\right)^{-1} \frac{d}{d\tau}(u^2)\bar{u} = -\bar{\nabla}\Phi. \tag{3.16}$$

Thus, the spacetime relativistic dynamical equations of motion in static spherically symmetric gravitational field can be obtained from (3.16). The time equation of motion is obtained as

$$a_{x_0} + \frac{1}{2c^2}\left(1 - \frac{u^2}{c^2}\right)^{-1} \frac{d}{d\tau}(u^2)u_{x^0} = 0 \tag{3.17}$$

or

$$\frac{d}{d\tau}\left[\left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \dot{x}^0\right] + \frac{1}{2c^2}\left(1 - \frac{u^2}{c^2}\right)^{-1} \frac{d}{d\tau}(u^2)u_{x^0} = 0. \tag{3.18}$$

Notice that the first term of equation (3.18) is exactly the expression obtained for the general relativistic time dilation and hence the second term is a correction term obtained from our dynamical approach in Schwarzschild's gravitational field.

Also, the respective azimuthal, polar and radial equations of motion are obtained as

$$\dot{r}\sin\theta\dot{\phi} + r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi} + \frac{1}{2c^2}\left(1 - \frac{u^2}{c^2}\right)^{-1}\frac{d}{d\tau}(u^2)u_\phi = 0, \tag{3.19}$$

$$r\ddot{\theta} + \dot{r}\dot{\theta} + \frac{1}{2c^2}\left(1 - \frac{u^2}{c^2}\right)^{-1}\frac{d}{d\tau}(u^2)u_\theta = 0 \tag{3.20}$$

and

$$a_r + \frac{1}{2c^2}\left(1 - \frac{u^2}{c^2}\right)^{-1}\frac{d}{d\tau}(u^2)u_r = -\frac{GM}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \tag{3.21}$$

with correction terms not found in the general relativistic approach.

4 Relativistic Dynamical Theory for Photons

The instantaneous passive and inertial mass of photons is given as

$$m_p = m_I = \frac{h\nu}{c^2}, \tag{4.1}$$

where h is Planck's constant. Precisely, as in Special Relativity, we postulate that the relativistic dynamical linear momentum of photons is given as

$$\overline{P} = \frac{h\nu}{c^2}\overline{u}, \tag{4.2}$$

where $\overline{u}$ is as defined in (3.2). The relativistic dynamical kinetic energy for photons is given as

$$T = (m_I - m_0)c^2 \tag{4.3}$$

or

$$T = h(\nu - \nu_0). \tag{4.4}$$

Also, as in Newton's dynamical theory of classical mechanics, the relativistic dynamical gravitational potential energy of photons(V_g) is postulated to be given by

$$V_g = m_p\Phi. \tag{4.5}$$

Hence, in static spherically symmetric gravitational fields

$$V_g = -\frac{h\nu}{c^2}\frac{GM}{r}. \tag{4.6}$$

Thus, the total mechanical energy E of a photon is given as

$$E = h(\nu - \nu_0) - \frac{h\nu}{c^2}\frac{GM}{r}. \tag{4.7}$$

If the mechanical energy of the photon is E_0 at $r = r_0$ then using the principle of conservation of mechanical energy it can be deduced that

$$\nu = \frac{E_0}{h}\left(1 - \frac{GM}{c^2 r}\right)^{-1} \tag{4.8}$$

or

$$\nu = \nu_0\left(1 - \frac{GM}{c^2 r_0}\right)\left(1 - \frac{GM}{c^2 r}\right)^{-1}. \tag{4.9}$$

Equation (4.9) is our newly derived expression for gravitational spectral shift for static spherically symmetric mass distributions with post Newtonian and post Einstein corrections of all orders of c^{-2}.

Also, the relativistic dynamical equation of motion for photons in static spherically symmetric gravitational fields can be obtained as

$$\frac{d}{d\tau}\left[\left(1 - \frac{GM}{c^2 r}\right)^{-1}\overline{u}\right] = -\left(1 - \frac{GM}{c^2 r}\right)^{-1}\overline{\nabla}\Phi \tag{4.10}$$

from which the instaneous velocity and acceleration vectors can be obtained.

5 Conclusion

Instructively, this approach unifies the dynamical and geometrical theories of gravitation for test particles and photons in static spherically symmetric gravitational fields. It is hoped that if it is well developed it can account for most corrections of theoretical results in gravitational fields. It is also hoped that this approach can also be used to establish the long desired unification of gravitational fields with other fundamental fields in nature.

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