# Quantum Uncertainty and Relativity

Sebastiano Tosto
Italy. Email: stosto@inwind.it

The major challenge of modern physics is to merge relativistic and quantum theories into a unique conceptual frame able to combine the basic statements of the former with the quantization, the non-locality and non-reality of the latter. A previous paper has shown that the statistical formulation of the space-time uncertainty allows to describe the quantum systems in agreement with these requirements of the quantum world. The present paper aims to extend the same theoretical model and approach also to the special and general relativity.

## 1 Introduction

Merging quantum mechanics and general relativity is surely the most challenging task of the modern physics. Since their early formulation these theories appeared intrinsically dissimilar, i.e. conceived for different purposes, rooted on a different conceptual background and based on a different mathematical formalism. It is necessary to clarify preliminarily what such a merging could actually mean.

A first attempt was carried out by Einstein himself in the famous EPR paper [1] aimed to bridge quantum behavior and relativistic constraints; he assumed the existence of hypothetical "hidden variables" that should overcome the asserted incompleteness of the quantum mechanics and emphasize the sought compatibility between the theories. Unfortunately this attempt was frustrated by successive experimental data excluding the existence of hidden variables. The subsequent development of both theories seemed to amplify further their initial dissimilarity; consider for instance the emergence of weird concepts like non-locality and non-reality of quantum mechanics, which make still more compelling the search of an unified view.

The most evident prerequisite of a unified model is the quantization of physical observables; being however the general relativity essentially a 4D classical theory in a curved non-Euclidean space-time, the sought model requires new hypotheses to introduce the quantization. A vast body of literature exists today on this topic; starting from these hypotheses several theories have been formulated in recent years, like the string theory [2,3] and loop quantum gravity [4], from which were further formulated the M-theory [5] and the supersimmetric theories [6]. The new way to represent the particles as vibrating strings and multi-dimensional branes is attracting but, even though consistent with the quantization, still under test. Moreover the quantization of the gravity field is not the only problem; additional features of the quantum world, the non-locality and non-reality, appear even more challenging as they make its rationale dissimilar from that of any other physical theory. The quantum mechanics postulates a set of mathematical rules based on the existence of a state vector $|\psi\rangle$ describing the quantum system in Hilbert space and a Hermitian operator corresponding to a measure, whose outcomes are the eigenvalues that represent the observables; the evolution of a system is represented by an evolution operator $T(t)$ such that $|\psi(t)\rangle = T(t)\,|\psi(0)\rangle$ operating on the state vector at the initial time. To these rules overlap also the exclusion and indistinguishability principles to formulate correctly the state vectors. The relativity rests on physical intuitions about the behavior of masses in a gravity field and in accelerated systems; it postulates the equivalence between gravitational and inertial mass and aims to build a covariant model of physical laws under transformation between inertial and non-inertial reference systems.

Apart from the apparent dissimilarity of their basic assumptions, a sort of conceptual asymmetry surely characterizes the quantum and relativistic theories; on the one side abstract mathematical rules, on the other side intuitive statements on the behavior of bodies in a gravity field. If the unification of these theories concerns first of all their basic principles, the task of introducing into a unified model even the concepts of non-locality and non-reality appears seemingly insurmountable. Eventually, a further concern involves the choice of the mathematical formalism appropriate to the unified approach. In general the mathematical formulation of any theoretical model is consequence of its basic assumptions. The tensor calculus is required to introduce covariant relativistic formulae in curvilinear reference systems; is however its deterministic character really suitable to formulate a non-real and non-local theoretical model? This last remark is suggested by previous papers that have already touched on this subject.

Early results showed that a theoretical approach based on the quantum uncertainty only, introduced as a unique assumption to calculate the electron energy levels of many-electron atoms/ions and diatomic molecules [7,8], could be subsequently extended to the special relativity too [9] while being also consistent with the concepts of non-localism and non-realism of quantum mechanics. Despite this encouraging background, however, so far the implications of the concepts introduced in the quoted papers have not been fully investigated and systematically exploited. In these early papers, the connection between quantum approach and special relativity was preliminarily acknowledged through gradual results progressively obtained, concerning however other less ambitious tasks; for instance, to assess the chance of superluminal speed of neutrinos [9]. The decisive strategy to this purpose was to regard the concept of uncertainty as a fundamental law of nature and not as a mere by-product of the commutation rules of operators. The statistical formulation of the quantum uncertainty has been proven effective on the one side to explain and account for all of the aforesaid features of the quantum world, i.e. quantization and non-reality and non-locality, and on the other side to obtain as corollaries the basic statements of special relativity too along with the invariant interval and Lorentz transformations. So it seemed sensible to exploit more profoundly these early achievements before proceeding towards a more advanced generalization including the general relativity too.

The present paper aims to collect together and push forward these preliminary results through further considerations having more general and systematic character; the approach proposed here is purposely focused towards a unifying task able to combine together quantum and relativistic requirements within the same conceptual frame. For this reason the present paper heavily rests on previous results introduced in the quoted references. While referring to the respective papers when necessary, some selected considerations very short and very important are again reported here for clarity of exposition and to make the present paper as self-contained as possible.

The paper consists of three parts. The first part, exposed in section 2, merely summarizes some concepts already published and some selected results previously achieved; these preliminary ideas are however enriched and merged together with new suggestions. The second part, section 3, stimulates further considerations approaching the intermediate target of merging together basic concepts of quantum mechanics and special relativity. The third part, section 4, aims to show that effectively even the most significant Einstein results of general relativity are compliant with the quantum approach here proposed.

The foremost concern constantly in mind is how to transfer into the beautiful self-consistency of relativity the alien concepts of quantization, non-locality and non-reality of the quantum world.

## 2 Preliminary considerations

The present section collects some ideas and results reported in previous papers concerning the statistical formulation of quantum uncertainty. Two equations sharing a common number of allowed states

$$\Delta x \Delta p_x = n\hbar = \Delta\varepsilon\Delta t \qquad (2,1)$$

are the only basic assumption of the present model. No hypothesis is made about size and analytical form of these ranges, which are by definition arbitrary. These equations disregard the local values of the dynamical variables, considered indeed random, unknown and unpredictable within their uncertainty ranges and thus of no physical interest. The concept of uncertainty requires the particle delocalized everywhere in its space range $\Delta x$ without any further detail about its actual motion; in practice the theoretical approach describes a system of quantum particles through their uncertainty ranges only exploiting the following positions

$$p_x \to \Delta p_x, \qquad x \to \Delta x, \qquad t \to \Delta t, \qquad \epsilon \to \Delta\epsilon. \qquad (2,2)$$

The first relevant consequence is that the calculations based on these ranges only waive in fact a specific kind of reference system. Consider for instance $\Delta x = x - x_o$: the lower boundary $x_o$ describes the position of $\Delta x$ with respect to the origin $O$ of an arbitrary reference system $R$, the upper boundary $x$ its size. So, owing to the lack of hypotheses or constraints on $x_o$ and $x$, the considerations inferred through the ranges (2,2) hold in any $R$ whatever it might be, Cartesian or curvilinear or else; also, being both boundary coordinates $x_o$ and $x$ arbitrary and unknowable, their role as concerns size and location of $\Delta x$ in $R$ could be identically exchanged. Hold also for the other ranges, e.g. for $t_o$ and $t$ of $\Delta t = t - t_o$, the same considerations introduced for $x_o$ and $x$, in particular the arbitrariness of the time coordinates in the reference system where is defined the time length $\Delta t$.

If in $R$ both boundaries are functions of time, as it is to be reasonably expected according to eqs. (2,1), then not only the range size is itself a function of time dependent on the relative signs and values of $\dot{x}$ and $\dot{x}_o$, but also the results hold for reference systems in reciprocal motion; indeed a reference system $R_o$ solidal with $x_o$ moves in $R$ at rate $\dot{x}_o$ and possible acceleration $\ddot{x}_o$. Nothing indeed compels to regard $\dot{x}_o$ as a constant, i.e. $R_o$ could be non-inertial or inertial depending on whether the concerned physical system admits or not accelerations. As any outcome inferred through the positions (2,2) holds by definition in an arbitrary reference system $R$ or $R_o$, it is clear since now the importance of this conclusion in relativity, which postulates covariant general laws of nature. Introducing local coordinates requires searching a covariant form for the physical laws thereafter inferred; once introducing arbitrary uncertainty ranges that systematically replace the local coordinates "a priori", i.e. conceptually and not as a sort of approximation, hold instead different considerations.

This topic will be concerned in the next subsection 4.1. Here we emphasize some consequences of the positions (2,2): (i) to waive a particular reference system, (ii) to fulfill the Heisenberg principle, (iii) to introduce the quantization through the arbitrary number $n$ of allowed states, (iv) to overcome the determinism of classical physics, (v) to fulfill the requirements of non-locality and non-reality [9]. Hence appears sensible to think that an approach based uniquely on eqs. (2,1) through the quantum positions (2,2) is in principle suitable to fulfil the requirements of special and general relativity too, far beyond the conceptual horizon of the quantum problems to which the quoted papers were early addressed. While being well known that the concept of uncertainty is a corollary of the operator formalism of wave mechanics, the reverse path is also possible: the operators of wave mechanics can be inferred from eqs. (2,1) [9]. The operator formalism is obtained introducing the probability $\Pi_x = \delta x/\Delta x$ for a free particle to be found in any sub-range $\delta x$ included in the whole $\Delta x$ during a given time range $\delta t$; it is only required that the sub-range be subjected to the same conditions of arbitrariness and uncertainty of $\Delta x$. Analogous considerations hold in defining the probability $\Pi_t = \delta t/\Delta t$ for the particle to be confined during a time sub-range $\delta t$ within a given $\delta x$, while $\Delta t$ is the time range for the particle to be within $\Delta x$. These probabilities allow to infer the operators

$$p_x \to \pm\frac{\hbar}{i}\frac{\partial}{\partial x}, \qquad \epsilon \to \pm\frac{\hbar}{i}\frac{\partial}{\partial t}. \tag{2,3}$$

As intuitively expected, the space and time sub-ranges $\delta x$ and $\delta t$ describe a wave packet having finite length and momentum that propagates through $\Delta x$ during $\Delta t$. The positions (2,2), directly related to eqs. (2,1), and the non-relativistic positions (2,3), inferred from eqs. (2,1), compare the two possible ways of introducing the quantum formalism. This result is important for two reasons: (i) it justifies why eqs. (2,1) lead to correct quantum results through the positions (2,2); (ii) the connection and consistency of the positions (2,2) with the familiar wave formalism (2,3) justifies the starting point of the present model, eqs. (2,1) only, as an admissible option rather than as an unfamiliar basic assumption to be accepted itself. Although both eqs. (2,1) and the wave equations introduce the delocalization of a particle in a given region of space, in fact the degree of physical information inherent the respective approaches is basically different: despite their conceptual link, eqs. (2,1) entail a degree of information lower than that of the wave formalism; hence they have expectedly a greater generality.

Consider a free particle. Eqs. (2,1) discard any information about the particle and in fact concern the delocalization ranges of its conjugate dynamical variables only; accordingly they merely acknowledge its spreading throughout the size of $\Delta x$ during the time uncertainty range $\Delta t$. Being also this latter arbitrary, the information provided by eqs. (2,1) concerns the number of states $n$ allowed to the particle and its average velocity component $v_x = \Delta x/\Delta t$ only. The wave mechanics concerns and describes instead explicitly the particle, which is regarded as a wave packet travelling throughout $\Delta x$; as it is known, this leads to the concept of probability density for the particle to be localized somewhere within $\Delta x$ at any time. The probabilistic point of view of the wave mechanics, consequence of $\Pi_x$ and $\Pi_t$, is replaced in eqs. (2,1) by the more agnostic total lack of information about local position and motion of the particle; this minimum information, consistent with the number of allowed states only, corresponds in fact to the maximum generality possible in describing the physical properties of the particle. The fact that according to eqs. (2,1) the particle could likewise be anywhere in all available delocalization range, agrees with the Aharonov-Bohm effect: the particle is anyhow affected by the electromagnetic field even in a region of zero field, because the probabilistic concept of "here and then there" is replaced by that of "anywhere" once regarding the region of the concerned field as a whole 3D uncertainty volume whose single sub-regions cannot be discerned separately. These conclusions also explain the so called "EPR paradox": the idea of spooky action at a distance is replaced by that of action at a spooky distance [9], because the positions (2,2) exclude the concept of local positions and thus that of a specific distance physically distinguishable from any other distance. Just because ignoring wholly and in principle the particle and any detail of its dynamics, while concerning instead uncertainty ranges only where *any* particle could be found, the indistinguishability of identical particles is already inherent the eqs. (2,1); instead it must be postulated in the standard quantum wave theory. The number $n$ of allowed states is the only way to describe the physical properties of the particle; this explains why $n$ plays in the formulae inferred from eqs. (2,1) the same role of the quantum numbers in the eigenvalues calculated solving the appropriate wave equations [7]. An evidence of this statement is shortly sketched for clarity in section 3.

The generality of eqs. (2,1) has relevant consequences: the approach based on these equations has been extended to the special relativity; instead the momentum and energy operators of eqs. (2,3) have limited worth being inherently non-relativistic. In effect the probabilities $\Pi_x$ and $\Pi_t$ have been inferred considering separately time and space; it was already emphasized in [9] that $\Pi_x$ and $\Pi_t$ should be merged appropriately into a unique space-time probability $\Pi(x, t)$. The necessity of a combined space-time reference system will be discussed in the next section 3. This fact suggests that a general description of the system is obtainable exploiting directly eqs. (2,1), which by their own definition introduce concurrently both space and time coordinates into the formulation of quantum problems; in short, the present paper upgrades the early concept of uncertainty to that of space-time uncertainty in the way highlighted below.

It has been shown that eqs. (2,1) also entail inherently the concepts of non-locality and non-reality of the quantum world: the observable outcome of a measurement process is actually the result of the interaction between test particle and observer, as a function of which early unrelated space and momentum ranges of the former collapse into smaller ranges actually related to $n$ according to eqs. (2,1); accordingly, it follows that the quantized eigenvalues are compliant with the non-locality and non-reality of quantum mechanics. This collapse is intuitively justified here noting that any measurement process aims to get information about physical observables;

without shrinking the initial unrelated ranges, thus reducing their degree of initial uncertainty, the concept of measurement would be itself an oxymoron. These results prospect therefore a positive expectation of relativistic generalization for the positions (2,2). Due to the subtle character of the connection between quantum and relativistic points of view, the present paper examines more closely in the next section the first consequences of the considerations just carried out, previously obtained in the quoted papers: the first goal to show the successful connection of eqs. (2,1) with the special relativity, is to infer the invariant interval and the Lorentz transformation.

## 3 Uncertainty and special relativity

The special relativity exploits 4-vectors and 4-tensors that consist of a set of dynamical variables fulfilling well defined transformation rules from one inertial reference system to another. For instance, the components $u_i$ of four velocity are defined by the 4-vector $dx_i$ as $u_i = dx_i(cdt)^{-1}(1-(v/c)^2)^{-1/2}$, being $v$ the ordinary 3D space velocity; the angular momentum is defined by the anti-symmetric 4-tensor $\mathrm{M}^{ik} = \sum(x^i p^k - x^k p^i)$, whose spatial components coincide with that of the vector $\mathbf{M} = \mathbf{r} \times \mathbf{p}$.

Despite the wealth of information available from such definitions, however, the central task always prominent in the present paper concerns their link to the concepts of quantization, non-locality and non-reality that inevitably qualify and testify the sought unification: if the final target is to merge quantum theory and relativity, seems ineffective to proceed on without a systematic check step after step on the compliance of such 4-vectors and tensors with the quantum world.

To explain in general the appropriate reasoning, compare the expectations available via tensor calculus and that available via the positions (2,2): having shown previously that eqs. (2,1) are compliant with the non-reality and non-locality, this means verifying the consistency of the former definitions of angular momentum or velocity with the concept of uncertainty. Since both of them necessarily exploit local coordinates, then, regardless of the specific physical problem to be solved, the previous definitions are in fact useless in the present model; the local coordinates are considered here worthless "a priori" in determining the properties of physical systems and thus disregarded.

Merging quantum and relativistic points of view compels instead to infer the angular momentum likewise as shown in [7], i.e. through its own physical definition via the positions (2,2) to exploit eqs. (2,1). For clarity this topic is sketched in the next sub-sections 3.4 to 3.7 aimed to show that indeed the well known relativistic expressions of momentum, energy and angular momentum of a free particle are inferred via trivial algebraic manipulations of eqs. (2,1) without exploiting the aforesaid standard definitions through local 4-coordinates.

Let us show now that the basic statements of special relativity are corollaries of eqs. (2,1) without any hypothesis on the uncertainty ranges. First, the previous section has shown that once accepting the positions (2,2) all inertial reference systems are indistinguishable because of the total arbitrariness of their boundary coordinates; if in particular both $x_o$ and $t_o$ are defined with respect to the origin of an inertial space-time reference system $R$, then the arbitrariness of the former require that of the latter. So in any approach based on eqs (2,1) only, all $R$ are necessarily equivalent in describing the eigenvalues, i.e. the observables of physical quantities. Second, it is immediate to realize that the average velocity $v_x = \Delta x/\Delta t$ previously introduced must be upper bounded. Consider a free particle in finite sized $\Delta x$ and $\Delta p_x$, thus with finite $n$; if $v_x \to \infty$ then $\Delta t \to 0$ would require $\Delta\varepsilon \to \infty$, which in turn would be consistent with $\varepsilon \to \infty$ as well. Yet this is impossible, because otherwise a free particle with finite local momentum $p_x$ could have in principle an infinite energy $\varepsilon$; hence, being by definition an allowed value of any physical quantity effectively liable to occur, the value of $v_x$ must be upper bound. Third, this upper value allowed to $v_x$, whatever its specific value might be, must be invariant in any inertial reference system. Indeed $v_x$ is defined in its own $R$ without contradicting the indistinguishability of all reference systems because its value is arbitrary like that of both $\Delta x$ and $\Delta t$; hence the lack of a definite value of $v_x$ lets $R$ indistinguishable with respect to other inertial reference systems $R'$ whose $v'_x$ is arbitrary as well. If however $v_x$ takes a specific value, called $c$ from now on, then this latter must be equal in any $R$ otherwise some particular $R^{(c)}$ could be distinguishable among any other $R'$, for instance because of the different rate with which a luminous signal propagates in either of them. Thus: finite and invariant value of $c$, arbitrariness of the boundary coordinates of $\Delta x$ and equivalence of all reference systems in describing the physical systems are strictly linked. One easily recognizes in these short remarks, straightforward corollaries of eqs (2,1), the basic statements of the special relativity.

This result legitimates thus the attempt to extend the outcomes of the non-relativistic approach of the early papers [7,8] to the special relativity. Before exemplifying some specific topics in the following subsections, it is useful to note that eqs. (2,1) can be read in several ways depending on how are handled the ranges in a given $R$.

The first example is provided by the ratio $\Delta x/\Delta t$: if the particle is regarded as a corpuscle of mass $m$ delocalized in $\Delta x$, thus randomly moving throughout this range, then $\Delta x/\Delta t$ is its average velocity component $v_x$ during $\Delta t$, whatever the local features of actual motion within $\Delta x$ might be. Interesting results can be inferred hereafter in a straightforward way. It is possible to define $\Delta p_x/\Delta t$ equal to $\Delta\varepsilon/\Delta x$ for any $n$, thus obtaining the concept of average force field component $F_x = \Delta p_x/\Delta t$ throughout $\Delta x$, or the related average power $\Delta\varepsilon/\Delta t = F_x v_x$ and so on. This is not mere dimensional exercise; these definitions hold without specifying a particular reference system and will be exploited in the following to check their ability to get both quantum and relativistic results.

In the next subsection will be examined in particular the ratio $\Delta p_x/\Delta x$ to introduce the curvature of the space-time simply via uncertainty ranges, i.e. in the frame of the uncertainty only. In these expressions, the ranges play the same role of the differentials in the respective classical definitions. This suggests how to regard the concept of derivative entirely in the frame of eqs. (2,1) only, i.e. as ratio of uncertainty ranges. The fact that the size of the ranges is arbitrary suggests the chance of thinking, for mere computational purposes, their limit sizes so small to exploit the previous definitions through the differential formalism; for instance it is possible to imagine a particle delocalized in a very small, but conceptually not vanishing, range $dx$ without contradicting any concept introduced in the positions (2,2), because remains valid in principle the statement $dx\Delta p_x = n\hbar$ despite the random values of $x$ between $x_o$ and $x_o + dx$ tend to the classical local value $x_o$. It is also possible to define very low values of $v_x$, i.e. $dx/\Delta t \ll c$, because $\Delta x$ and $\Delta t$ are independent ranges and so on. Furthermore, hypothesizing $\hbar$ so small that all ranges can be even treated as differentials, let us try to regard and handle the ranges of eqs. (2,1) as if in the limit case $n = 1$ they would read $(dx)(dp_x) = \hbar = (dt)(d\varepsilon)$. This means that, for mere computational purposes, the case $n = 1$ is regarded as a boundary condition to be fulfilled when calculating the sought physical property.

To check the validity of this point through an example of calculation involving $v_x$, rewrite eqs. (2,1) in the forms $\Delta p_x/\Delta t = \Delta\varepsilon/\Delta x$ and $\Delta\varepsilon = \Delta p_x \Delta x/\Delta t$ that however will be now handled likewise as if $dp_x/dt = F_x = d\varepsilon/dx$ and $d\varepsilon = v_x dp_x$ to assess the results hereafter obtainable. In agreement with these computational notations, which however do not mean at all regarding the formal position $\Delta x/\Delta t \to dx/dt$ as a local limit, let us consider a free particle and write

$$\varepsilon = \int v'_x (dp_x/dv'_x) dv'_x. \tag{3,1}$$

Although these positions are here introduced for calculation purposes only, since actually the uncertainty ranges are by definition incompatible with the concept of differential limit size tends to zero, nevertheless it is easy to check their validity recalling that in a previous paper [9] simple considerations based on eqs. (2,1) only allowed to infer $p_x = \varepsilon v_x/c^2$; this equation is so important that its further demonstration based on a different reasoning is also provided below in subsection 3.4. Replacing in eq (3,1) and integrating yields $\varepsilon = c^{-2}\int v'_x[d(\varepsilon v'_x)/dv'_x]dv'_x$, easily solved in closed form; the solution $\varepsilon = const\left(1-(v_x/c)^2\right)^{-1/2}$ yields by consequence also $p_x = v_x c^{-2} const\left(1-(v_x/c)^2\right)^{-1/2}$. If $v_x \to 0$ then $p_x \to 0$; yet nothing compels also the vanishing of $\varepsilon$. Calculating thus the limit $p_x/v_x$ for $v_x \to 0$ and calling $m$ this finite limit,

$$\lim_{v_x\to 0}\frac{p_x}{v_x} = m, \tag{3,2}$$

one infers the integration constant $const = \pm mc^2$; follow immediately the well known expressions

$$p_x = \pm m v_x\left(1-(v_x/c)^2\right)^{-1/2},$$
$$\varepsilon = \pm mc^2\left(1-(v_x/c)^2\right)^{-1/2}. \tag{3,3}$$

The double sign corresponds in the former case to that of either velocity component, in the latter case to the existence of antimatter. Moreover exploit also $\Delta p_x/\Delta t - \Delta\varepsilon/\Delta x = 0$; regarding again this equation in its computational differential form $dp_x/dt - d\varepsilon/dx = 0$ and solving it with respect to $v_x$, as if the ranges would really be differentials, one finds of course $v_x = -\Delta x/\Delta t$. These results are important: handling the ranges as differentials entails just the well known relativistic results, which appear however to be limit cases i.e. boundary conditions of the respective definitions via uncertainty ranges; this confirms that the intervals appearing in the invariant interval and in the Lorentz transformation of length and time must be actually regarded as uncertainty ranges, as pointed out in [9], so that also the transformation formulae get full quantum meaning. This holds provided that the ranges related to $\hbar$ be really so small with respect to distances and times of interest to justify the integral calculus; this is certainly true in typical relativistic problems that usually concern massive bodies or cosmological distances and times.

So far the particle has been regarded as a corpuscle characterized by a mass $m$ traveling throughout $\Delta x$ during the time range $\Delta t$. According to the positions (2,3) and owing to the results [9], however, the particle can be identically described as a wave propagating throughout the same space range during the same time range; also to this purpose are enough eqs. (2,1), the basic assumptions of the wave formalism are unnecessary.

Let us regard $\Delta x$ as the space range corresponding to one wavelength and the related $\Delta t$ as a reciprocal frequency $\omega = \Delta t^{-1}$; so one finds $\Delta\varepsilon = n\hbar\omega$ with $\omega = 2\pi\nu$, in which case $\Delta x/\Delta t = \omega\lambda = v$ as well. In principle one expects from this result that in general an average velocity $v_1$ corresponds to the frequency $\omega_1$, thus $v_2$ to $\omega_2$ and so on. Suppose that, for fixed $\Delta x$, a time range $\Delta t'$ and thus a frequency $\omega'$ exist such that the right hand side turns into a unique constant velocity, whose physical meaning will appear soon; then, using again the differential formalism, $d(\lambda^{-1}) = -\lambda^{-2}d\lambda$ and $\lambda d\omega' + \omega' d\lambda = 0$ combined into $\lambda(d\omega' - \lambda\omega' d(\lambda^{-1})) = 0$ yield $v'/2\pi = d\omega'/dk$ where $k = 2\pi/\lambda$. Being $v'$ arbitrary like $\Delta x$, including the trivial factor $2\pi$ in $v'' = v'/2\pi$ yields $v'' = d\omega'/dk$. So are defined the phase and group velocities $v$ and $v''$ of a wave, which of course coincide if $v$ does not depend on $\omega$; this is possible because $\Delta x$ and $\Delta t$ are independent ranges that can fulfil or not this last particular case. Moreover eqs. (2,1) also yield immediately $\Delta\varepsilon/\Delta p = d\nu/d(\lambda^{-1}) = v$. Eventually, dividing both sides of $\Delta x\Delta p_x = n\hbar$ by $\Delta t$ yields

$F\Delta x = n\hbar\omega$; since $dF/dv$ has physical dimension of momentum, being all range sizes arbitrary the last equation reads in general $p = h/\lambda$. These reasonable results are distinctive features of quantum mechanics, here found as corollaries by trivial manipulations of eqs. (2,1). If both corpuscle and wave formalisms are obtained from a unique starting point, eqs. (2,1), then one must accept the corpuscle/wave dual behavior of particles, as already inferred in [9]. This justifies why these equations have been successfully exploited in the early papers [7,8] to describe the quantum systems.

After having checked the compliance of eqs. (2,1) with the fundamental principles of both quantum mechanics and special relativity, we are now justified to proceed further towards the connection between the theories. Eqs. (2,1) allow describing various properties of quantum systems, e.g. in the frame of space/time uncertainty or energy/momentum uncertainty, as better specified in the next subsection. Note that the invariant interval, inferred itself from eqs. (2,1) only, is compliant with the non-locality and non-reality simply regarding the space and time intervals as uncertainty ranges; by consequence merging quantum mechanics and special relativity simply requires abandoning the deterministic meaning of intervals defined by local coordinates, which have classical character and thus are exactly known in principle. Indeed we show below that the invariant interval consists of ranges having fully quantum meaning of space-time uncertainty. In the frame of eqs. (2,1) only, the concept of time derivative necessarily involves the time uncertainty range; an example is $\Delta x/\Delta t$ previously identified with the velocity $v_x$. This latter, even though handled as $dx/dt$ for computational purposes only, still keeps however its physical meaning of average velocity.

These considerations hold in the reference system $R$ where are defined eqs. (2,1) and suggest a remark on the algebraic formalism; once trusting on eqs. (2,1) only, the concept of derivative is replaced by that of ratio between uncertainty ranges. These latter indeed represent the chance of variability of local quantities; so the derivative takes here the meaning of correlation between these allowed chances. Of course being the ranges arbitrary and unknown, this chance is extended also to the usual computational concept of derivative, as shown before. Once having introduced through the uncertainty the requirements of quantum non-locality and non-reality into the relativistic formulae, a problem seems arising at this point, i.e. that of the covariancy.

This point will be concerned in the next section 4, aimed to discuss the transformations between inertial and non-inertial reference systems. For clarity of exposition, however, it is better to continue the present introductory discussion trusting to the results so far exposed; it is enough to anticipate here that the arbitrariness of the quantum range boundaries, and thus that of the related reference systems as well, is the key topic to merge the requirements of uncertainty and covariancy.

### 3.1 The space-time uncertainty

This section aims to show that the concept of space-time is straightforward corollary of the space/momentum and time/energy uncertainties. Eqs. (2,1) represent the general way of correlating the concepts of space, momentum, time and energy by linking their uncertainties through the number $n$ of allowed states; just their merging defines indeed the eigenvalues of any physical observable. On the one side, therefore, the necessity of considering concurrently both time and space coordinates with analogous physical meaning appears because of the correlation of their uncertainties; for instance the particular link underlying time and space ranges through $c$ allows to infer the invariant interval and the relativistic expressions of momentum and energy. On the other side the concept of quantization appears strictly related to that of space-time, since the concurrence of both $\Delta x$ and $\Delta t$ that defines $n$ also introduces in fact a unique space-time uncertainty. These elementary considerations highlight the common root between relativity and quantum theory, which also accounts for the non-locality and non-reality of the latter according to the conclusions emphasized in [9].

Eqs. (2,1) consist of two equations that link four ranges; for any $n$, two of them play the role of independent variables and determine a constrain for the other two, regarded therefore as dependent variables. In principle this means that two independent ranges introduce eqs. (2,1) via $n$. As $\Delta p_x$ and $\Delta\varepsilon$ include local values of physical observables while $\Delta x$ and $\Delta t$ include local values of dynamical variables, it is reasonable to regard as a first instance just these latter as arbitrary independent variables to which are related momentum and energy as dependent variables for any $n$; however any other choice of independent variables would be in principle identically admissible.

For instance, let us concern $\Delta\varepsilon\Delta x/(v_x/c) = n\hbar c$ considering fixed the energy and coordinate ranges. Two limits of this equation are particularly interesting: (i) $v_x/c \to 0$, which requires in turn $n \to \infty$, and (ii) $v_x \lesssim c$, which requires $\Delta x \lesssim n\hbar c/\Delta\varepsilon$ for any given $n$. Consider the former limit rewriting identically $(\Delta p_x/v_x)v_x\Delta x = n\hbar$, which reads $v_x\Delta x\Delta m = n\hbar$ according to eq (3,2); since for a free particle $v_x$ is a constant, then $\Delta(mv_x) = \Delta p_x$ i.e. $p_x \approx mv_x$. Guess the related classical energy regarding again $\Delta\varepsilon/\Delta p_x = v_x$ as $d\varepsilon/dp_x = v_x$, whence $d\varepsilon = v_x m dv_x$ i.e. $\varepsilon = mv_x^2/2 + const$. As expected, these expressions of energy and momentum result to be just the non-relativistic limits of eqs. (3,3) for $v_x \ll c$. This is because we have considered here the space coordinate separately from the time coordinate: despite the time range has been somehow introduced into the previous reasoning through the definition of $v_x$, yet it occurred in the way typical of the Newtonian mechanics, i.e. regarding the time as an entity separated from the space coordinate, and not through the link between $\Delta p_x$ and $\Delta\varepsilon$ provided by $n$.

We also know that the classical physics corresponds to the limit $n \rightarrow \infty$ [9]; thus eqs. (2,1) require that the non-relativistic limit $v_x \ll c$ and the classical physics limit $n \rightarrow \infty$ are actually correlated. Indeed, eqs. (3,3) have been obtained handling the ranges as differentials just thanks to small values of $n$. Of course such a correlation is not required when regarding quantum theory and relativity separately, it appears instead here as a consequence of their merging. Since for $n \rightarrow \infty$ the difference between $n$ and $n+1$ becomes more and more negligible with respect to $n$, this latter tends to behave more and more like a continuous variable. It has been shown in [9] that just the quantization entails the non-real and non-local features of the quantum world; instead locality and reality are asymptotic limit properties of the classical world attained by the continuous variable condition $n \rightarrow \infty$. Now it appears that just the same quantization condition of $n$ requires also the relativistic properties of the particles, which indeed are well approximated by the corresponding equations of Newtonian physics in the limit $n \rightarrow \infty$ i.e. $v_x \ll c$. Otherwise stated, the special relativity rests itself on the quantization condition required by the space/momentum and time/energy uncertainties merged together; these latter are therefore the sought unique fundamental concept on which are rooted quantum properties, non-reality, non-locality and special relativity.

### 3.2 Energy-momentum uncertainty and Maxwell equations

Let us start from $\Delta\varepsilon = v_x \Delta p_x$; being as usual $\Delta\varepsilon = \varepsilon - \varepsilon_o$ and $\Delta p_x = p_x - p_o$, this uncertainty equation splits into two equations $\varepsilon = v_x p_x$ and $\varepsilon_o = v_x p_o$ defined by the arbitrary boundary values of energy and momentum. Consider first the former equation; dividing both sides by an arbitrary volume V and by an arbitrary velocity component $v_x^{\S}$, the uncertainty equation turns dimensionally into the definition $J_x^{\S} = C^{\S} v_x$ of a mass flow; indeed $J_x^{\S}$ is the flux of the mass $m$ initially defining momentum and energy of the particle, $C^{\S}$ is the corresponding amount of mass per unit volume. Calculating the flux change between any $x$ and $x + \delta x$ during $\delta t$, one finds $\delta J_x^{\S} = v_x \delta C^{\S} + C^{\S} \delta v_x$. This result can be exploited in various ways. For instance in a previous paper it has been shown that eqs. (2,1) lead under appropriate hypotheses to the result $J_x^{\S} = -D\partial C^{\S}/\partial x$ [10], being $D$ the diffusion coefficient of $m$. The particular case of constant $v_x$ in the absence of an external force field acting on $m$ during the time range $\delta t = \delta x/v_x$ yields $\delta J_x^{\S} = -[\partial(D\partial C^{\S}/\partial x)/\partial x]\delta x$. Since $\delta J_x^{\S}/\delta x = -\delta C^{\S}/\delta t$, because $\delta J_x^{\S}/\delta x$ and $\delta C^{\S}/\delta t$ have opposite sign under the hypothesis of gradient driven mass flow in the absence of sinks or sources in the diffusing medium, one obtains the 1D Fick law $\delta C^{\S}/\delta t = \partial(D\partial C^{\S}/\partial x)/\partial x$, trivially extensible to the 3D case. In general, under the constrain of constant $v_x$ only, the vector equations corresponding to $J_x^{\S} = C^{\S} v_x$ and $\delta J_x^{\S} = -v_x \delta C^{\S}$ read

$$\mathbf{J}^{\S} = C^{\S}\mathbf{v}, \qquad \nabla \cdot \mathbf{J}^{\S} = -\partial C^{\S}/\partial t. \tag{3,4}$$

Multiplying by $e/m$ both sides of these expressions, one obtains the corresponding equations for the flux of charge density $C_e$, i.e. $\mathbf{J}_e = C_e \mathbf{v}$. An analogous result holds for the second part $\varepsilon_o = v_x p_o$ of the initial uncertainty equation, rewritten now as $\mathbf{J}_m = C_m \mathbf{v}$ with $C_m = C^{\S} e_m/m$; the physical meaning of $e_m$ will be remarked below. Put now $C = C_e + C_m$ and $\mathbf{J} = \mathbf{J}_e + \mathbf{J}_m$; then, replacing $\mathbf{J}^{\S}$ and $C^{\S}$ of the mass concentration gradient equation with $\mathbf{J}$ and $C$, it is possible to introduce an arbitrary vector $\mathbf{U}_-$ such that the second equation eq (3,4) reads

$$\nabla \cdot \nabla \times \mathbf{U}_- = \nabla \cdot \mathbf{J} + \frac{\partial C}{\partial t} \tag{3,5}$$

as it is clear because the left hand side is null. So one obtains

$$\nabla \times \mathbf{U}_- = \frac{\partial \mathbf{U}_+}{\partial t} + \mathbf{J}, \qquad \nabla \cdot \mathbf{U}_+ = C,$$
$$\mathbf{J} = \mathbf{J}_e + \mathbf{J}_m, \qquad C = C_e + C_m. \tag{3,6}$$

The second equation defines $\mathbf{U}_+$. Since $C = C_e + C_m$, the vector $\mathbf{U}_+$ must reasonably have the form $\mathbf{U}_+ = \mathbf{H} + \mathbf{E}$, where $\mathbf{H}$ and $\mathbf{E}$ are arbitrary vectors to be defined. As also $\mathbf{J}$ is sum of two vectors, $\mathbf{U}_-$ is expected to be itself sum of two vectors too. For mere convenience let us define these latter again through the same $\mathbf{H}$ and $\mathbf{E}$; there is no compelling reason to introduce necessarily further vectors about which additional hypotheses would be necessary to solve the first eq (3,6). Appears now sensible to guess $\mathbf{U}_- = c(\mathbf{H} - \mathbf{E})$, with $c$ mere dimensional factor, for four reasons: (i) $\mathbf{U}_+ + c^{-1}\mathbf{U}_- = 2\mathbf{H}$ and $\mathbf{U}_+ - c^{-1}\mathbf{U}_- = 2\mathbf{E}$, i.e. $\mathbf{U}_-$ and $\mathbf{U}_+$ can be expressed through the same vectors they introduce; (ii) the same holds for the scalars $c^{-1}\mathbf{U}_+ \cdot \mathbf{U}_- = H^2 - E^2$ and $U_+^2 - c^{-2}U_-^2 = 4\mathbf{E} \cdot \mathbf{H}$; (iii) the same holds also for $c^{-1}\mathbf{U}_- \times \mathbf{U}_+ = 2\mathbf{E} \times \mathbf{H}$ and (iv) $U_+^2 + c^{-2}U_-^2 = 2(H^2 + E^2)$. If $\mathbf{H}$ and $\mathbf{E}$ are now specified in particular as vectors proportional to magnetic and electric fields, then the proposed definitions of $\mathbf{U}_-$ and $\mathbf{U}_+$ entail a self-consistent set of scalars and vectors having some interesting features: the scalars (ii) define two invariants with respect to Lorentz transformations, whereas the vector (iii) is proportional to the Poynting vector and defines the energy density flux; moreover the point (iv) defines a scalar proportional to the energy density of the electromagnetic field; eventually the integral $\int \mathbf{U}_+ \cdot \mathbf{U}_- d\mathrm{V}$ over the volume previously introduced is proportional to the Lagrangian of a free field.

Although eqs. (3,5) and (3,6) are general equations straightforward consequences of charge flows, simply specifying purposely them to the case of the electromagnetic field follows the validity of the form assigned to $\mathbf{U}_-$ because of such sensible outcomes. The first eq (3,6) reads thus $c\nabla\times(\mathbf{H}-\mathbf{E}) = \partial(\mathbf{H} + \mathbf{E})/\partial t + (\mathbf{J}_e + \mathbf{J}_m)$. In principle the terms of this equation containing $\mathbf{H}$, $\mathbf{E}$, $\mathbf{J}_e$ and $\mathbf{J}_m$ can be associated in various ways, for instance is admissible $c\nabla \times \mathbf{H} = \partial\mathbf{H}/\partial t + \mathbf{J}_m$; integrating this equation is certainly possible but the solution $\mathbf{H} = \mathbf{H}(x, y, z, t, \mathbf{J}_m)$ would be of scarce interest, i.e. one would merely find the space and time profile of a possible

$\mathbf{H}$ consistent with $\mathbf{J}_m$. The same would hold considering the analogous equation for $\mathbf{E}$. A combination of mixed terms that appears more interesting is

$$\nabla\cdot\mathbf{E}=C_e, \qquad \nabla\cdot\mathbf{H}=C_m,$$
$$-c\nabla\times\mathbf{E}=\frac{\partial\mathbf{H}}{\partial t}+\mathbf{J}_m, \qquad c\nabla\times\mathbf{H}=\frac{\partial\mathbf{E}}{\partial t}+\mathbf{J}_e. \tag{3,7}$$

In this form, the interdependence of the magnetic and electric field vectors $\mathbf{H}$ and $\mathbf{E}$ through $\mathbf{J}_e$ and $\mathbf{J}_m$ yields the Maxwell equations formulated in terms of charge and current densities. These equations, also inferred from eqs. (2,1), have been written having in mind the maximum generality; $C_e$ and $C_m$ are proportional to the electric charge and magnetic charge densities, $\mathbf{J}_e$ and $\mathbf{J}_m$ to the charge and magnetic current densities. While $C_e$ is known, an analogous physical meaning for $C_m$ is doubtful because the magnetic "monopoles" are today hypothesized only but never experimentally observed. Although it is certainly possible to regard these equations with $C_m = 0$ and $\mathbf{J}_m = 0$, nevertheless seems formally attractive the symmetric character of the four equations (3,7). Note however in this respect that rewriting $\mathbf{E} = \mathbf{E}_o + \mathbf{Q}$ and $\mathbf{H} = \mathbf{H}_o + \mathbf{W}$, where $\mathbf{W}$ and $\mathbf{Q}$ are further field vectors whose physical meaning is to be defined, with the positions

$$C'_e=\nabla\cdot\mathbf{Q}, \qquad \nabla\times\mathbf{Q}=0, \qquad \mathbf{J'}_e=\frac{\partial\mathbf{Q}}{\partial t},$$
$$\rho_m=-\nabla\cdot\mathbf{W}, \qquad \nabla\times\mathbf{W}=0, \qquad \mathbf{J'}_m=\frac{\partial\mathbf{W}}{\partial t},$$

the equations (3,7) turn into

$$\nabla\cdot\mathbf{E}_o=C_e-C'_e, \qquad \nabla\cdot\mathbf{H}_o=\rho_m,$$
$$-c\nabla\times\mathbf{E}_o=\frac{\partial\mathbf{H}_o}{\partial t}+\mathbf{J'}_m, \qquad c\nabla\times\mathbf{H}_o=\frac{\partial\mathbf{E}_o}{\partial t}-\mathbf{J'}_e+\mathbf{J}_e,$$

having put here $C_m = 0$ and $\mathbf{J}_m = 0$. In practice rewriting $\mathbf{H}$ and $\mathbf{E}$ as a sum of vectors $\mathbf{H}_o$ and $\mathbf{E}_o$ parallel to them plus $\mathbf{W}$ and $\mathbf{Q}$ fulfilling the aforesaid conditions one obtains a new set of Maxwell equations whose form, even without reference to the supposed magnetic monopoles, is however still the same as if these latter would really exist. Note eventually that beside eqs. (3,7) there is a further non-trivial way to mix the electric and magnetic terms, i.e.

$$\nabla\cdot\mathbf{E}=C_e, \qquad \nabla\cdot\mathbf{H}=C_m,$$
$$-c\nabla\times\mathbf{E}=\frac{\partial\mathbf{H}}{\partial t}+\mathbf{J}_e, \qquad c\nabla\times\mathbf{H}=\frac{\partial\mathbf{E}}{\partial t}+\mathbf{J}_m, \tag{3,8}$$

expectedly to be read with $C_m = 0$ and $\mathbf{J}_m = 0$. Work is in progress to highlight the possible physical meaning of $\mathbf{Q}$ and $\mathbf{W}$ and that of the eqs. (3,8) still consistent with eq (3,6).

### 3.3 Uncertainty and wave formalism

Start now from eqs. (3,3) that yield $\varepsilon^2 = (cp_x)^2 + (mc^2)^2$; so the positions (2,3) define the known 2D Klein-Gordon equation $-\partial^2\psi_o/c^2\partial t^2 = -\partial^2\psi_o/\partial x^2+(mc/\hbar)^2\psi_o$, whose extension to the 4D case is trivial simply assuming $\psi_o = \psi_o(x, y, z, t)$

$$\frac{\partial^2\psi_o}{c^2\partial t^2}-\nabla^2\psi_o+k^2\psi_o=0, \qquad k^2=\left(\frac{mc}{\hbar}\right)^2. \tag{3,9}$$

Eq. (3,9) is equivalent to $O_5^2\psi_o = 0$ inferred from $\mathbf{O}_5\psi_o = 0$, where the total momentum operator $\mathbf{O}_5$ is defined as

$$\mathbf{O}_5=\mathbf{a}_j\frac{\hbar}{i}\frac{\partial}{\partial x_j}+\mathbf{a}_4\frac{i}{c}\frac{\hbar}{i}\frac{\partial}{\partial t}+\mathbf{a}_5 mc,$$
$$j=1,2,3; \qquad \mathbf{a}_j\cdot\mathbf{a}_{j'}=\delta_{j,j'}.$$

Thus $\mathbf{O}_5$ is the sought linear combination $\mathbf{a}_j\mathrm{P}_j+(\mathbf{a}_4 i/c)\mathrm{H}+\mathbf{a}_5 mc$ of the momentum $\mathrm{P}_j$ and energy $\mathrm{H}$ operators (2,3) via orthogonal unit vector coefficients $\mathbf{a}_j$ and $\mathbf{a}_4 i/c$ and $\mathbf{a}_5$; this combination of space and time operators defines the wave equation corresponding to the relativistic eqs. (3,3).

Replace now $\psi_o$ with $\psi = \psi_o+\mathbf{a}\cdot\mathbf{A}+b\varphi$ in eq (3,9); $\mathbf{a}$ and $b$ are arbitrary constants, $\mathbf{A}$ and $\varphi$ are functions of $x_j, t$ that must still fulfill eq (3,9). Assuming constant both modulus and direction of $\mathbf{a}$ with respect to $\mathbf{A}$, trivial calculations yield three equations. One is once again the Klein-Gordon equation for $\psi_o$; moreover subtracting and summing to the two remainder terms the amount $\mathbf{a}\cdot\mathbf{J}/c$, where $\mathbf{J}$ is a further arbitrary vector, the condition $\mathbf{a}\cdot\mathbf{J}/cb = -\rho$ yields the following two equations

$$\frac{\partial^2\varphi}{c^2\partial t^2}-\nabla^2\varphi+k^2\varphi-\rho=0,$$
$$\frac{\partial^2\mathbf{A}}{c^2\partial t^2}-\nabla^2\mathbf{A}+k^2\mathbf{A}-\frac{\mathbf{J}}{c}=0. \tag{3,10}$$

In principle this result is anyway formally possible with the given $b$, which links the equations through $\rho$ and $\mathbf{J} = \rho\mathbf{v}$ according to eqs. (3,4). The condition on $b$ requires $\mathbf{a}\cdot\mathbf{J}/c\rho = \mathbf{a}'\cdot\mathbf{J}'/c\rho'$; so in general $\mathbf{J}$ is not necessarily a constant. Let us specify now this result. If $\mathbf{A}$ and $\varphi$ are proportional to the magnetic and electric potentials, then $\rho$ and $\mathbf{J}$ are charge density and flux; in effect the particular case $\varphi \propto r^{-1}$ agrees with the physical meaning of the former, whence the meaning of the latter as well. The fact that $\psi_o$ differs from $\psi = \psi_o + \mathbf{a}\cdot(\mathbf{A}-\mathbf{J}\varphi/c\rho)$ by the vector $\mathbf{A}-\mathbf{J}\varphi/c\rho \neq 0$ suggests defining $\mathbf{a} = \xi\mathbf{J}'/c$ in order obtain the scalar $\mathbf{J}'\cdot\mathbf{A}/c - \varphi\mathbf{J}'\cdot\mathbf{J}/\rho c^2$, i.e. $\mathbf{J}'\cdot\mathbf{A}/c - \rho'\varphi\mathbf{v}'\cdot\mathbf{v}/c^2$; $\xi$ is a proportionality factor. So putting $\varphi = \varphi' q$, with $q$ proportionality factor, the result is $\mathbf{J}'\cdot\mathbf{A}/c - \rho'\varphi'$ with $q^{-1} = \mathbf{v}'\cdot\mathbf{v}/c^2$. In this way one obtains $\psi = \psi_o + \xi(\mathbf{J}'\cdot\mathbf{A}/c - \rho'\varphi')$, while eqs. (3,10) are the well known Proca's equations in vector form.

Note that $\xi$ has physical dimension $field^{-2}$, which indeed justifies the particular way of defining $\mathbf{a}$, while the scalar in parenthesis characterizes the wave function of a particle moving in the presence of magnetic and electric potentials.

Since a free particle has by definition kinetic energy only, the scalar additive to $\psi_o$ is a perturbative term due to the magnetic and electric potentials; so it should reasonably represent a kinetic energy perturbation due to the presence of magnetic and electric fields. This suggests that the complete Lagrangian $T - U$ of the particle moving in the electromagnetic field should be therefore given by the linear combination of the scalar just found and the free field scalar $c\mathbf{U}_- \cdot \mathbf{U}_+ = H^2 - E^2$, i.e. it should be obtained by volume integration of $\mathbf{J}' \cdot \mathbf{A}/c - \rho'\varphi' + \chi(E^2 - H^2)$, being $\chi$ an appropriate coefficient of the linear combination of potential and kinetic energy terms.

This topic is well known and does not deserve further comments. It is worth noticing instead that eqs. (3,10) can be also obtained introducing the extended space-time momentum operator $\mathbf{O}_7$ collecting together the space and time operators of the positions (2,3) in a unique linear combination expressed as follows

$$\mathbf{O}_7 = \mathbf{a}_j\partial/\partial x_j + \mathbf{a}_4 i\partial/\partial(ct) + \mathbf{a}_5 i/x_5 + \mathbf{a}_6\partial/\partial x_6 + \mathbf{a}_7\partial/\partial x_7,$$

where $x_5$, $x_6$ and $x_7$ are to be regarded as extra-coordinates. Putting $x_5 = \hbar/mc$, the wave function that yields directly both eqs. (3,10) with this operator reads accordingly

$$\psi = \psi_o + \mathbf{a} \cdot (\mathbf{A} - \mathbf{J}x_5^2/c)x_6 + (\varphi - \rho x_5^2)x_7.$$

Still holds the position $\mathbf{a}_j \cdot \mathbf{a}_{j'} = \delta_{j,j'}$ that regards again the various $\mathbf{a}_j$, with $j = 1..7$, as a set of orthogonal unit vectors in a 7D dimensional space where is defined the equation $O_7^2\psi = 0$ containing as a particular case the Klein-Gordon equation. The sixth and seventh addends of $\mathbf{O}_7$ are ineffective when calculating $O_7^2\psi_o$, which indeed still yields the free particle equation; however just these addends introduce the non-null terms of Proca's equations in the presence of fields.

In summary, the free particle eq (3,9) is nothing else but the combination of the two eqs. (3,3) expressed through the wave formalism of quantum mechanics; its successive manipulation leads to define the Lagrangian of the electromagnetic field in the presence of magnetic and electric potentials while introducing additional extra-dimensions. It appears however that the chance of defining 3 extra-dimensions to the familiar ones defining the space-time is suggested, but not required in the present model, by the relativistic wave formalism only.

### 3.4 Uncertainty and invariant interval

In [9] has been inferred the following expression of invariant interval

$$\Delta x^2 - c^2\Delta t^2 = \delta s^2 = \Delta x'^2 - c^2\Delta t'^2 \qquad (3,11)$$

in two inertial reference systems $R$ and $R'$. Owing to the fundamental importance of this invariant in special relativity, from which can be inferred the Lorentz transformations [11], we propose here a further instructive proof of eq (3,11) based uniquely on the invariance of $c$. Consider then the uncertainty range $\Delta x = x - x_o$ and examine how its size might change during a time range $\Delta t$ if in general $x = x(\Delta t)$ and $x_o = x_o(\Delta t)$.

Let be $\delta_\pm = \Delta x \pm v\Delta t$ the range in $R$ that generalizes the definition $\Delta x/\Delta t = v_x$ to $\delta_\pm \neq 0$ through a new velocity component $v \neq v_x$ taking also into account the possible signs of $v$. Regard both $\delta_\pm$ as possible size changes of $\Delta x$ during the time range $\Delta t$ in two ways: either (i) with $x_o$ replaced by $x_o \pm v\Delta t$ while keeping fixed $x$ or (ii) with $x$ replaced by $x \pm v\Delta t$ while keeping fixed $x_o$. Of course the chances (i) or (ii) are equivalent because of the lack of hypotheses on $\Delta x$ and on its boundary coordinates. In both cases one finds indeed $\delta_+ = \Delta x + v\Delta t$ and $\delta_- = \Delta x - v\Delta t$, which yield $\overline{\delta} = (\delta_+ + \delta_-)/2 = \Delta x$; so the range size $\Delta x$, seemingly steady in $R$, is actually a mean value resulting from random displacements of its lower or upper boundaries from $x_o$ or $x$ at average rates $v = \dot{x}_o$ or $v = \dot{x}$ as a function of time. Of course $v$ is in general arbitrary. The actual space-time character of the uncertainty, hidden in $\overline{\delta}$, appears instead explicitly in the geometric mean $< \delta >=< \delta_-\delta_+ >= (\Delta x^2 - v^2\Delta t^2)^{1/2}$ of both time deformations allowed to $\Delta x$. Note however that the origin $O$ of the reference system $R$ where is defined $\Delta x$ appears stationary in (ii) to an observer sitting on $x_o$ because is $x$ that displaces, but in (i) $O$ appears moving to this observer at rate $\mp\dot{x}_o$. Consider another reference system $R'$ solidal with $x_o$, thus moving in $R$ at rates $\pm\dot{x}_o$. In $R'$ is applicable the chance (ii) only, as $x_o$ is constant; it coincides with the origin in $R'$ and, although it does not in $R$, yet anyway $\dot{x}_o = 0$. So the requirement that both (i) and (ii) must be equivalent to describe the deformation of $\Delta x$ in $R$ and $R'$, otherwise these reference systems would be distinguishable, requires concluding that the chance (ii) must identically hold itself both in $R$ and $R'$. This is possible replacing $v = \dot{x} = c$ in $< \delta >$, which indeed makes in this particular case the deformation rate (ii) of $\Delta x$ indistinguishable in $R$ and $R'$: in both systems $\dot{x}_o = 0$, as $x_o$ is by definition constant, whereas $\dot{x}$ also coincides because of the invariancy of $c$; when defined through this particular position, therefore, $< \delta^{(c)} >$ is invariant in any $R$ and $R'$ in agreement with eqs. (3,11). These equations have been written considering spacelike intervals; of course an identical reasoning holds also writing eqs. (3,11) as timelike intervals.

### 3.5 The invariancy of eqs. (2,1)

The following considerations concern the invariancy of eqs. (2,1) in different inertial reference systems. The proof is based on the arbitrariness of the range sizes and on the fact that in any $R$ and $R'$ actually $n$ is indistinguishable from $n'$ pertinent to the different range sizes resulting from the Lorentz transformations; indeed neither $n$ nor $n'$ are specified and specifiable by assigned values, rather they symbolize arbitrary numbers of states. Admitting different range sizes in inertial reference systems in reciprocal motion, the chance of any $n$ in $R$ corresponds to any other chance allowed to $n'$ in $R'$. However the fact that the ranges are arbitrary compels considering the totality of values of $n$ and $n'$, not their single values, in agreement with the physical meaning of eqs. (2,1). Hence, despite the individual numbers of states can be different for specific $\Delta x\Delta p_x$ in $R$ and $\Delta x'\Delta p'_x$ in $R'$, the sets of all arbitrary integers represented by all $n$ and $n'$ remain in principle indistinguishable regardless of how any particular $n$ might transform into another particular $n'$.

The fact of having inferred in [9] the interval invariant in inertial reference systems, the Lorentz transformations of time and length and the expression $p_x = \varepsilon v_x/c^2$, should be itself a persuasive proof of the compliance of eqs. (2,1) with special relativity; now it is easy to confirm this conclusion demonstrating the expression of $p_x$ in a more straightforward way, i.e. exploiting uniquely the concept of invariancy of $c$. The present reasoning starts requiring an invariant link between $\Delta p_x = p_1 - p_o$ and $\Delta\varepsilon = \varepsilon_1 - \varepsilon_o$ in $\Delta\varepsilon = \Delta p_x\Delta x/\Delta t$. This is possible if $\Delta x/\Delta t = c$, hence $\Delta p_x c = \Delta\varepsilon$ is a sensible result: it means of course that any local value $\varepsilon$ within $\Delta\varepsilon$ must be equal to $cp_x$ calculated through the corresponding local value $p_x$ within $\Delta p_x$ although both are unknown. If however $\Delta x/\Delta t < c$, the fact that the arbitrary $v_x$ is not an invariant compels considering for instance $v_x^k\Delta x/\Delta t = qc^{k+1}$ with $k$ arbitrary exponent and $q < 1$ arbitrary constant. Then $(\Delta p_x v_x^{-k})c^{k+1}q = \Delta\varepsilon$ provides in general an invariant link of $\Delta p_x v_x^{-k}$ with $\Delta\varepsilon$ through $c^{k+1}q$. Is mostly interesting the chance $k = 1$ that makes the last equation also consistent with the previous particular case, i.e. $(\Delta p_x/v_x)c^2q = \Delta\varepsilon$; so one finds $\varepsilon_1 v'_x/c^2 - p_1 = \varepsilon_o v'_x/c^2 - p_o$ with $v'_x = v_x/q$. The arbitrary factor $q$ is inessential because $v_x$ is arbitrary itself, so it can be omitted; hence $p_x = \varepsilon v_x/c^2$ when considering any local values within the respective ranges because of the arbitrariness of $p_o$, $p_1$, $\varepsilon_o$, $\varepsilon_1$. At this point holds identically the reasoning of the previous subsection. Rewrite $\Delta\varepsilon - (\Delta p_x/v_x)c^2 = 0$ as $\delta_\pm = \Delta\varepsilon \pm (\Delta p_x/v)c^2 \neq 0$ with $v \neq v_x$ to calculate $\overline{\delta} = \Delta\varepsilon$ and $<\delta\varepsilon> = \pm\sqrt{\Delta\varepsilon^2 - (\Delta p_x/v)^2c^4}$; one concludes directly that the invariant quantity of interest is that with $v = c$, i.e. $\delta\varepsilon_c = \pm\sqrt{\Delta\varepsilon^2 - \Delta p_x^2c^2}$ that reads

$$\Delta\varepsilon^2 = \delta\varepsilon_c^2 + \Delta p_x^2c^2. \tag{3,12}$$

So $\varepsilon^2 = (mc^2)^2 + p_x^2c^2$ once having specified $\delta\varepsilon_c$ with the help of eq (3,2). This is not a trivial way to obtain again eqs (3,3). In general the ranges are defined by arbitrary boundary values; then $\varepsilon_1$ and $\varepsilon_o$ can be thought in particular as arbitrary values of $\varepsilon$, thus invariant themselves if calculated by means of eqs. (3,3). So, despite the local values within their own uncertainty ranges are unknown, the range $\Delta\varepsilon$ defined as the difference of two invariant quantities must be invariant itself. Consider thus in particular the interval of eq (3,11). It is interesting to rewrite this result with the help of eqs. (2,1) as $(n\hbar)^2\Delta p_x^{-2} - c^2(n\hbar)^2\Delta\varepsilon^{-2} = \delta s^2 = \Delta x'^2 - c^2\Delta t'^2$, which yields therefore

$$\delta p_x\delta s = n\hbar = \delta p'_x\delta s, \tag{3,13}$$

$$\delta p_x = \pm\frac{\Delta p_x\Delta\varepsilon}{\sqrt{\Delta\varepsilon^2 - (c\Delta p_x)^2}}, \qquad \delta p'_x = \pm\frac{\Delta p'_x\Delta\varepsilon'}{\sqrt{\Delta\varepsilon'^2 - (c\Delta p'_x)^2}}.$$

So $\delta p_x = \delta p_x(\Delta p_x, \Delta\varepsilon)$, whereas $\delta p'_x = \delta p'_x(\Delta p'_x, \Delta\varepsilon')$ as well. Both $\delta s$ and $\delta p_x$ at left hand side are invariant: the former by definition, the latter because formed by quantities $\Delta\varepsilon$ and $\Delta p_x$ defined by invariant boundary quantities $\varepsilon_1$, $\varepsilon_o$, $p_1$, $p_o$ of the eqs. (3,3). Being the range sizes arbitrary and not specifiable in the present theoretical model, the first eq. (3,13) is nothing else but the first eqs. (2,1) explicitly rewritten twice with different notation in invariant form. This feature of the first eq. (3,13) confirms not only the previous reasoning on $n$ and $n'$, thus supporting the relativistic validity of eqs. (2,1) in different inertial reference systems, but also the necessity of regarding the ranges of special relativity as uncertainty ranges; in other words the concept of invariancy merges with that of total arbitrariness of $n$, on which was based the previous reasoning. In conclusion: (i) disregarding the local coordinates while introducing the respective uncertainty ranges according to the positions (2,2) is enough to plug the classical physics into the quantum world; (ii) replacing the concepts of space uncertainty and time uncertainty with that of space-time uncertainty turns the non-relativistic quantum physics into the relativistic quantum physics; (iii) the conceptual step (ii) is fulfilled simply considering time dependent range sizes; (iv) if the deterministic intervals of special relativity are regarded as uncertainty ranges, then the well known formulae of special relativity are in fact quantum formulae that, as a consequence of eqs. (2,1), also fulfil the requirements of non-locality and non-reality. Accordingly, it is not surprising that the basic postulates of special relativity are in fact corollaries of eqs. (2,1) only, without the need of any further hypothesis.

### 3.6 The angular momentum

Let us show how the invariant interval of eq (3,11) leads to the relativistic angular momentum. Expand in series the range $\delta s = \sqrt{\Delta x^2 - c^2\Delta t^2}$ noting that in general

$$\sqrt{a^2 - b^2} = a - \left(b/a + (b/a)^3/4 + (b/a)^5/8 + \cdot\cdot\right)b\big/2.$$

Calculated with an arbitrary number of terms, the series expansion can be regarded as an exact result. Thus write $\delta s = \delta r_x - \delta r_t/2$ where $\delta r_t = c\Delta t\left[c\Delta t/\Delta x + (c\Delta t/\Delta x)^3/4 + \cdot\cdot\right]$ and $\delta r_x = \Delta x$. Being $\Delta t$ and $\Delta x$ both arbitrary, $\delta r_x$ and $\delta r_t$ are independent ranges. Regard $\delta s$ as the $x$-component of an arbitrary uncertainty vector range $\delta\mathbf{s} = \delta\mathbf{r}_s - \delta\mathbf{r}_t/2$ and repeat identically the reasoning introduced in [7] and shortly sketched here; the subscripts stand for "space" and "time". Insert $\delta\mathbf{s}$ in the classical component $M_w = \delta\mathbf{s}\times\delta\mathbf{p}\cdot\mathbf{w}$ of angular momentum $\mathbf{M}$ along the arbitrary unit vector $\mathbf{w}$. The analytical form of the function expressing the local value $\mathbf{p}$ does not need to be specified; according to the positions (2,2) $\mathbf{p}$ is a random value to be replaced by its own uncertainty range $\delta\mathbf{p}$ to find the eigenvalues of the angular momentum. For the mere fact of having introduced an invariant interval into the definition of angular momentum, therefore, $M_w$ results defined by the sum of two scalars $M_{w,s} = \delta\mathbf{r}_s \times \delta\mathbf{p} \cdot \mathbf{w}$ and $M_{w,t} = -\delta\mathbf{r}_t \times \delta\mathbf{p} \cdot \mathbf{w}/2$. So $M_{w,s} = \mathbf{w} \times \delta\mathbf{r}_s \cdot \delta\mathbf{p}$, i.e. $M_{w,s} = \delta\mathbf{p} \cdot \delta\mathbf{I}_s$ with $\delta\mathbf{I}_s = \mathbf{w} \times \delta\mathbf{r}_s$. If $\delta\mathbf{p}$ and $\delta\mathbf{I}_s$ are orthogonal then $M_{w,s} = 0$; else $M_{w,s} = \delta p_{I_s} \delta I_s$, defined by the conjugate dynamical variables $\delta p_{I_s} = \delta\mathbf{p} \cdot \delta\mathbf{I}_s / |\delta\mathbf{I}_s|$ and $\delta I_s = |\delta\mathbf{I}_s|$, yields immediately by virtue of eqs. (2,1) $M_{w,s} = \pm l\hbar$ with $l$ arbitrary integer including zero; instead of $n$, we have used the standard notation $l$ for the eigenvalues of angular motion of the particle. Identically one finds also $M_{w,t} = \pm l'\hbar/2$, with $l'$ arbitrary integer including zero too. Hence $M_w = \pm l\hbar \pm l'\hbar/2$.

The first addend is clearly the non-relativistic component $l\hbar$ of angular momentum already found in [7], the latter yields an additional component $l'\hbar/2$ of angular momentum. Having considered the invariant range $\delta s$ rather than the space range $\Delta x$ only, the further number $l'$ of states is due to the time term of the space-time uncertainty; putting $\Delta t = 0$, i.e. omitting the time/energy uncertainty and thus the time coordinate, $\delta r_t = 0$ and $M_w$ coincides with the non-relativistic quantum component of angular momentum only.

Four important remarks concern:

(i) the number $l$ of states allowed for the non-relativistic angular momentum component coincides with the quantum number of the eigenvalue of the non-relativistic angular momentum wave equation;

(ii) the concept of space-time uncertainty defines the series development of the particular invariant range $\delta s$ as sum of two terms, the second of which introduces a new non-classical component of angular momentum $l'/2$;

(iii) the local momentum $\mathbf{p}$ and local coordinate $\mathbf{s}$ within the ranges $\delta\mathbf{p}$ and $\delta\mathbf{s}$ are not really calculated, rather they are simply required to change randomly within the respective ranges of values undetermined themselves; (iv) the boundary coordinates of both $\delta\mathbf{p}$ and $\delta\mathbf{s}$ do not appear in the result, rather is essential the concept of delocalization ranges only to infer the total component as a sum of both eigenvalues.

The component $M_w = \pm l\hbar \pm s\hbar$, with $s = l'/2$, requires introducing $\mathbf{M} = \mathbf{L} + \mathbf{S}$. In [7] the non-relativistic $M_{nr}^2$ has been calculated summing its squared average components between arbitrary values $-L$ and $+L$ allowed for $\pm l$, with $L$ by definition positive, thus obtaining $M_{nr}^2 = 3 < (\hbar l)^2 >= L(L+1)\hbar^2$. Replace now $\pm l$ with $\pm l \pm s$; with $j = l \pm s$ ranging between arbitrary $-J$ and $J$, then $M^2 = 3 < (\hbar j)^2 >= 3(2J+1)^{-1} \sum_{-J}^{J} (\hbar j)^2 = \hbar^2 J(J+1)$ being $J$ positive by definition. The obvious identity $\sum_{-J}^{J} j^2 \equiv 2 \sum_{0}^{J} j^2$ requires that $J$ consistent with $M^2$ takes all values allowed to $|j|$ from $|l-s|$ up to $|l+s|$ with $l \leq L$ and $s \leq S$. Since no hypothesis has been made on $\mathbf{L}$ and $\mathbf{S}$, this result yields in general the addition rule of quantum vectors. Also, holds for $\mathbf{S}$ the same reasoning carried out for $\mathbf{L}$ in [7], i.e. only one component of $\mathbf{S}$ is known, whereas it is immediate to realize that $S^2 = \hbar^2(L'/2+1)L'/2$.

The physical meaning of $\mathbf{S}$ appears considering that: (i) $l'\hbar/2$ is an angular momentum, inferred likewise as and contextually to $l\hbar$; (ii) $l'$ results when considering the invariant space-time uncertainty range into the definition of $M_w$; (iii) $l$ and $l'$ are independent, indeed they concern two independent uncertainty equations; the former is related to the angular motion of the particle, the latter must be instead an intrinsic property of the particle, as $l'$ is defined regardless of whether $l = 0$ or $l \neq 0$. Since in particular $l' \neq 0$ even though the orbital angular momentum is null, $\mathbf{S}$ can be nothing else but the intrinsic property of the particle we call spin angular momentum. Indeed it could be also inferred in the typical way of reasoning of the special relativity i.e. introducing observers and physical quantities in two different inertial reference systems $R$ and $R'$ in relative constant motion; so, exploiting exactly the same procedure considering couples $\delta\mathbf{r}$ and $\delta\mathbf{p}$ together with $\delta\mathbf{r}'$ and $\delta\mathbf{p}'$ fulfilling the Lorentz transformation one finds of course the same result.

It is significant the fact that here the spin is inferred through the invariant interval of eq (3,13), i.e. exploiting eqs. (2,1) only. This is another check of the conceptual compliance of these equations with the special relativity.

### 3.7 The hydrogenlike atom/ion

The following example of calculation concerns first the non-relativistic hydrogenlike atom/ion. Assume first the origin $O$ of $R$ on the nucleus, the energy is thus $\varepsilon = p^2/2m - Ze^2/r$ being $m$ the electron mass. Since $p^2 = p_r^2 + M^2/r^2$, the positions (2,2) $p_r \to \Delta p_r$ and $r \to \Delta r$ yield $\varepsilon = \Delta p_r^2/2m + M^2/2m\Delta r^2 - Ze^2/\Delta r$. Two numbers of states, i.e. two quantum numbers, are expected because of the radial and angular uncertainties. Eqs. (2,1) and the results of section 3.3 yield $\varepsilon = n^2\hbar^2/2m\Delta r^2 + l(l+1)\hbar^2/2m\Delta r^2 - Ze^2/\Delta r$ that reads $\varepsilon = \varepsilon_o + l(l+1)\hbar^2/2m\Delta r^2 - E_o/n^2$ with $E_o = Z^2e^4m/2\hbar^2$ and $\varepsilon_o = (n\hbar/\Delta r - Ze^2m/n\hbar)^2/2m$. Minimize $\varepsilon$ putting $\varepsilon_o = 0$, which yields $\Delta r = n^2\hbar^2/Ze^2m$ and $\varepsilon_{tot} = [l(l+1)/n^2 - 1]E_o/n^2$; so $l \leq n-1$ in order to get $\varepsilon < 0$, i.e. a bound state. Putting thus $n = n_o + l + 1$ one finds the electron energy levels $\varepsilon_{el} = -E_o/(n_o+l+1)^2$ and the rotational energy $\varepsilon_{rot} = l(l+1)E_o/n^4$ of the atom/ion as a whole around $O$. So $\varepsilon_{rot} = \varepsilon_{tot} - \varepsilon_{el}$. Repeat the same reasoning putting $O$ on the center of mass of the system nucleus + electron; it is trivial to infer $E'_o = Z^2e^4m_r/2\hbar^2$ and $\Delta r' = n^2\hbar^2/Ze^2m_r$, being $m_r$ the electron-nucleus reduced mass. If instead $O$ is fixed on the electron, i.e. the nucleus moves with respect to this latter, then $E''_o = Z^2e^4A/2\hbar^2$ and $\Delta r'' = n^2\hbar^2/Ze^2A$, being $A$ the mass of the nucleus. Thus various reference systems yield the same formula, and then again $\varepsilon'_{rot} = \varepsilon'_{tot} - \varepsilon'_{el}$ and $\varepsilon''_{rot} = \varepsilon''_{tot} - \varepsilon''_{el}$, yet as if the numerical result would concern particles of different mass.

The ambiguity between change of reference system and change of kind of particle is of course only apparent; it depends merely on the erroneous attempt of transferring to the quantum world dominated by the uncertainty the classical way of figuring an "orbital" system of charges where one of them really rotates around the other. Actually the uncertainty prevents such a phenomenological way of thinking: instead the correct idea is that exists a charge located somewhere with respect to the nucleus and interacting with it, without chance of specifying anything else. This is shown noting that anyway one finds $E_{el} = -Ze^2/2\Delta\rho$ with $\Delta\rho$ symbolizing any radial range of allowed distances between the charges, regardless of which particle is actually in $O$. Since the total uncertainty range $2\Delta\rho$ is the diameter of a sphere centered on $O$, the different energies are mere consequence of different delocalization extents of a unique particle with respect to any given reference point.

This reasoning shows that different ranges of allowed radial momenta entail different allowed energies: if the particle of mass $m$ is replaced for instance by one of lower mass, then $\Delta\rho$ increases while therefore $\Delta p_\rho$ decreases; i.e. $E_o$ reasonably decreases along with the range of allowed radial momenta. Of course it is not possible to infer "a priori" if these outcomes concern the motion of three different particles or the motion of a unique particle in three different reference systems; indeed no specific mass appears in the last conclusion. The allowed radial momenta only determine $\varepsilon_{el}$, defined as $-E_o$ of two charges $-Ze$ and $e$ at diametric distance with respect to $O$ times $n^{-2}$; this latter is the fingerprint of the quantum delocalization meaning of $\Delta\rho$. So $E_o$ is defined by the mass $m$ of the particle whose energy levels are of interest; for instance in the case of a mesic atom $m$ would be the mass of a negative muon.

Note that $\varepsilon_{el}$ is the intrinsic energy of the system of two charges, regardless of the kinetic energy of the atom as a whole and the rotational energy, i.e. $\Delta\varepsilon = \varepsilon_{tot} - \varepsilon_{el} = l(l+1)E_o/n^2$. The physical meaning of the boundary coordinates of $\Delta x$ and $\Delta t$ has been already emphasized.

Let us consider now the boundary values of other uncertainty ranges, examining also the harmonic oscillator and the angular momentum. The vibrational and zero point energies of the former $n\hbar\omega$ and $\hbar\omega/2$ define $\Delta\varepsilon = \varepsilon_{tot} - \varepsilon_{zp} = n\hbar\omega$; i.e. the lower boundary of the range is related to an intrinsic energy not due to the oscillation of the mass, likewise as that of the hydrogenlike atom was the binding energy. In the case of angular momentum $\Delta M_w = M_w - l'\hbar = l\hbar$, with $M_w \equiv M_{tot,w}$, i.e. the lower boundary of the range is still related to the intrinsic angular momentum component of the particle; from this viewpoint, therefore, the spin is understandable as the intrinsic property not dependent on the specific state of motion of the particle with respect to which the arbitrary values of $l$ define the range size $\Delta M_w$. The same holds for the relativistic kinetic energy of a free particle; the series development of the first eq (3,3) shows that its total energy is the rest energy plus higher order terms, i.e. one expects $\Delta\varepsilon = \varepsilon - mc^2$; also now the lower boundary of the range is an intrinsic feature of the particle, not related to its current state of motion. Classically, the energy is defined an arbitrary constant apart; here it appears that this constant is actually an intrinsic property of the particle, not simply a mathematical requirement, and that a similar conclusion should hold in general, thus expectedly also for the relativistic hydrogenlike energy. Let us concern the relativistic case specifying the energy ranges in order to infer the binding energy $\varepsilon_{el} < 0$ through trivial manipulations of eq (3,12) $\Delta\varepsilon^2 = c^2\Delta p^2 + \delta\varepsilon_c^2$. This expression is the 4D extension of that considering the component $\Delta p_x$ only; whatever the three space components and their link to $\Delta p$ might be, their arbitrariness allows to write again $\Delta p = p_1 - p_o$ and $\Delta\varepsilon = \varepsilon_1 - \varepsilon_o$. The first steps of calculations are truly trivial: consider $c\Delta p/\delta\varepsilon_c$ then calculate $(c\Delta p - \Delta\varepsilon)/\delta\varepsilon_c$, so that $(cp_1 - \Delta\varepsilon)/\delta\varepsilon_c = b + \sqrt{a^2-1} - a$ with $a = \Delta\varepsilon/\delta\varepsilon_c$ and $b = cp_o/\delta\varepsilon_c$. Next $(cp_1 - \Delta\varepsilon)^2/\delta\varepsilon_c^2$ yields trivially

$$\frac{\Delta\varepsilon^2}{(cp_1-\Delta\varepsilon)^2} - \frac{(c\Delta p)^2}{(cp_1-\Delta\varepsilon)^2} = \frac{1}{\left(b+\sqrt{a^2-1}-a\right)^2}.$$

A reasonable position is now $(cp_1 - \Delta\varepsilon)^2 = (c\Delta p)^2$: indeed the left hand side $\Delta\varepsilon^2/(c\Delta p)^2 = 1$ for $b \to \infty$, i.e. for $\delta\varepsilon_c \to 0$, agrees with the initial equation. Trivial manipulations yield

$$\frac{cp_1}{\Delta\varepsilon} = 1 \pm \frac{1}{\sqrt{1+\left(b+\sqrt{a^2-1}-a\right)^{-2}}},$$

$$c\Delta p = \pm(cp_1 - \Delta\varepsilon), \qquad a = \frac{\Delta\varepsilon}{\delta\varepsilon_c}, \qquad b = \frac{cp_o}{\delta\varepsilon_c}.$$

This result has not yet a specific physical meaning because it has been obtained simply manipulating the ranges $\Delta\varepsilon$, $\delta\varepsilon_c$ and $c\Delta p$. Physical information is now introduced taking the minus sign and calculating the non-vanishing first order term of series development of the right hand side around $b = \infty$, which is $1/2b^2$; the idea that specifies the result is thus the non-relativistic hydrogenlike energy $-(\alpha Z/n)^2 mc^2/2$ previously found. Requiring $b = n/\alpha Z$, the limit of the ratio $cp_1/\Delta\varepsilon$ is thus the energy in $mc^2$ units gained by the electron in the bound state with respect to the free state. To infer $a$ recall that $n = l+1$ and note that the second equation $\pm\Delta\varepsilon = cp_o - cp_1 \pm cp_1$ reads $\pm\Delta\varepsilon = cp_o$ or $\pm\Delta\varepsilon = cp_o - 2cp_1$; dividing both sides by $\delta\varepsilon_c$, the latter suggests $cp_1/\delta\varepsilon_c = (2\alpha Z)^{-1}$ in order that $\pm a = n/\alpha Z$ or $\pm a = (n-1)/\alpha Z$ read respectively $\pm a = (l+1)/\alpha Z$ or $\pm a = l/\alpha Z$, i.e. $a = (l + 1/2 \pm 1/2)/\alpha Z$.

In conclusion the relativistic form of the binding energy $\varepsilon_{el}$ is

$$\frac{\varepsilon_{el}}{mc^2} = \sqrt{1+\frac{(\alpha Z)^2}{\left(n+\sqrt{(j+1/2)^2-1}-(j+1/2)\right)^2}} - 1$$

with $j = l \pm s$. If $n \to \infty$ then $\varepsilon_{el} \to 0$, while the non-relativistic limit previously found corresponds to $\alpha Z \to 0$.

### 3.8 The pillars of quantum mechanics

Let us show now that the number of allowed states introduced in eqs. (2,1) leads directly to both quantum principles of exclusion and indistinguishability of identical particles. The results of the previous section suggest the existence of different kinds of particles characterized by their own values of $l'$. If this conclusion is correct, then the behavior of the particles should depend on their own $l'$. Let us consider separately either possibility that $l'$ is odd or even including 0.

If $l'/2$ is zero or integer, any change of the number $N$ of particles is physically indistinguishable in the phase space: are indeed indistinguishable the sums $\sum_{j=1}^{N} l_j + Nl'/2$ and $\sum_{j=1}^{N+1} l_j^* + (N+1)l'/2$ controlling the total value of $M_w$ before and after increasing the number of particles; indeed the respective $l_j$ and $l_j^*$ of the $j$-th particles are arbitrary. In other words, even after adding one particle to the system, $M_w$ and thus $M^2$ replicate any possible value allowed to the particles already present in the system simply through a different assignment of the respective $l_j$; so, in general a given number of allowed states determining $M_w$ in not uniquely related to a specific number of particles.

The conclusion is different if $l'$ is odd and $l'/2$ half-integer; the states of the phase space are not longer indistinguishable with respect to the addition of particles since $M_w$ jumps from ...integer, half-integer, integer... values upon addition of each further particle, as any change of the number of particles necessarily gives a total component of $M_w$, and then a resulting quantum state, different from the previous one. In other words any odd-$l'$ particle added to the system entails a new quantum state distinguishable from those previously existing, then necessarily different from that of the other particles. The conclusion is that a unique quantum state is consistent with an arbitrary number of even-$l'$ particles, whereas a unique quantum state characterizes each odd-$l'$ particle. This is nothing else but a different way to express the Pauli exclusion principle, which is thus corollary itself of quantum uncertainty. Recall also the corollary of indistinguishability of identical particles, already remarked; eqs. (2,1) concern neither the quantum numbers of the particles themselves nor their local dynamical variables but ranges where *any* particle could be found, whence the indistinguishability.

We have shown that a unique formalism based on eqs. (2,1) only is enough to find the basic principles of both special relativity and quantum mechanics; also, quantum and relativistic results have been concurrently inferred. The only essential requirement to merge special relativity and quantum mechanics is to regard the intervals of the former as the uncertainty ranges of the latter. The next step concerns of course the general relativity.

## 4 Uncertainty and general relativity

In section 3 the attempt to generalize the non-relativistic results of the papers [7,8] was legitimated by the possibility of obtaining preliminarily the basic postulates of special relativity as straightforward corollaries of eqs. (2,1). Doing so, the positions (2,2) ensure that the special relativity is compliant with the concepts of quantization, non-reality and non-locality of quantum mechanics [9]. At this point, the attempt of extending further an analogous approach to the general relativity is now justified by showing two fundamental corollaries: (i) the equivalence of gravitational and inertial forces and (ii) the coincidence of inertial and gravitational mass. These concepts, preliminarily introduced in [9], are so important to deserve being sketched again here.

Once accepting eqs. (2,1) as the unique assumption of the present model, the time dependence of the uncertainty range sizes $\Delta x = x - x_o$ and $\Delta p_x = p_x - p_o$ rests on their link to $\Delta t$ through $n$; for instance it is possible to write $d\Delta x/d\Delta t$ in any $R$ without contradicting eqs. (2,1); this position simply means that changing $\Delta t$, e.g. the time length allowed for a given event to be completed, the space extent $\Delta x$ necessary for the occurring of that event in general changes as well. In other words there is no reason to exclude that $\Delta t \to \Delta t + \Delta t^{\S}$, with $\Delta t^{\S}$ arbitrary, affects the sizes of $\Delta x$ and $\Delta p_x$ although $n$ remains constant; in fact eqs. (2,1) do not prevent such a possibility. Hence, recalling that here the derivative is the ratio of two uncertainty ranges, the rate $\Delta\dot{x}$ with which changes $\Delta x$ comes from the chance of assuming $\dot{x} = \delta x/d\Delta t$ and/or $\dot{x}_o = \delta x_o/d\Delta t$; also, since analogous considerations hold for $d\Delta p_x/d\Delta t$ one finds similarly $\dot{p}_x$ and $\dot{p}_o$. Also recall that the boundary values of the ranges are arbitrary, so neither $p_o$ and $p_x$ nor their time derivatives need to be specified by means of assigned values. Since $\dot{p}_o$ and $\dot{p}_x$ are here simply definitions, introduced in principle but in fact never calculated, the explicit analytical form of the momentum $p$ of general relativity does not need to be known; the previous examples of angular momentum and hydrogenlike atoms elucidate this point. The following reasoning exploits therefore the mere fact that a local force is related to a local momentum change, despite neither the former nor the latter are actually calculable functions of coordinates.

Let us define $\Delta t$ and the size change rates $d\Delta x/d\Delta t$ and $d\Delta p_x/d\Delta t$ in an arbitrary reference system $R$ as follows

$$d\Delta p_x/d\Delta t = F = -n\hbar\Delta x^{-2} d\Delta x/d\Delta t \qquad (4,1)$$

with $F \neq 0$ provided that $\dot{x} \neq \dot{x}_o$ and $\dot{p}_x \neq \dot{p}_o$. At left hand side of eqs. (4,1) the force component $F$ involves explicitly the mass of the particle through the change rate of its momentum; at the right hand side $F$ concerns the range $\Delta x$ and its size change rate only, while the concept of mass is implicitly inherent the physical dimensions of $\hbar$. It is easy to explain why a force field arises when changing the size of $\Delta x$: this means indeed modifying also the related size of $\Delta p_x$ and thus the extent of values allowed to the random $p_x$; the force field is due to the resulting $\dot{p}_x$ throughout $\Delta x$ whenever its size is altered. After having acknowledged the link between $\Delta\dot{x}$ and

$F$ intuitively suggested by eqs. (2,1), the next task is to check the conceptual worth of eqs. (4,1). Let $x_o$ be the coordinate defined with respect to the origin $O$ of $R$ where hold eqs. (2,1). If $\Delta t = t - t_o$ with $t_o = const$, then the previous expression reads $d\Delta p_x/dt = F = -n\hbar\Delta x^{-2}d\Delta x/dt$. Formally eqs. (4,1) can be rewritten in two ways depending on whether $x_o$ or $x$, and likewise $p_o$ or $p_x$, are considered constants: either (i) $\Delta\dot{p}_x \equiv \dot{p}_x$ so that $\dot{p}_x = F_x = -n\hbar\Delta x^{-2}\dot{x}$ or (ii) $\Delta\dot{p}_x \equiv \dot{p}_o$ so that $\dot{p}_o = F_o = -n\hbar\Delta x^{-2}\dot{x}_o$.

The physical meaning of these results is realized imagining in $R$ the system observer + particle: the former is sitting on $x_o$, the latter is fixed on $x$. In (i) the observer is at rest with respect to $O$ and sees the particle accelerating according to $\dot{p}_x$ by effect of $F_x$ generated in $R$ during the deformation of the space-time range $\Delta x$. In (ii) the situation is different: now $\Delta x$ deforms while also moving in $R$ at rate $\dot{x}_o$ with respect to $O$, the deformation occurs indeed just because the particle is at rest with respect to $O$; thus the force $F_o$ displaces the observer sitting on $x_o$, which accelerates with respect to the particle and to $O$ according to $-\dot{p}_o$. In a reference system $R_o$ solidal with $x_o$, therefore, a force $F'_o$ still acts on the observer although he is at rest; the reason is clearly that $R_o$ is non-inertial with respect to $R$ because of its local acceleration related to $-\dot{p}_o$. Although the reasoning is trivially simple, the consequence is important: both situations take place in the presence of a force component because both cases (i) and (ii) are equally allowed and conceptually equivalent; however the force in $R$ is real, it accelerates a mass, that in $R_o$ does not; yet $F_x \neq 0$ compels admitting in $R$ also $F_o \neq 0$, which in turn reads $F'_o \neq 0$ in $R_o$. Whatever the transformation rule from $F_o$ to $F'_o$ might be, the conclusion is that an observer in an accelerated reference frame experiences a force similar to that able to accelerate a massive particle with respect to the observer at rest. Of course $F_x$ is actually the component of a *force field*, because it is an average value defined throughout a finite sized range $\Delta x$ deforming as a function of time, whereas $F_o$ and $F'_o$ are by definition *local* forces in $x_o$; if however the size of $\Delta x$ is smaller and smaller, then $F_x$ is better and better defined itself like a classical local force.

Now we are also ready to find the equivalence between inertial and gravitational mass. Note indeed that $F_x$ has been defined through a unique mass $m$ only, that appearing in the expression of momentum; hence from the standpoint of the left hand side of eqs. (4,1) we call $m$ inertial mass. Consider in this respect that just this mass must somehow appear also at right hand side of eqs. (4,1) consisting of uncertainty ranges only, which justifies the necessary position $n\hbar\Delta\dot{x}\Delta x^{-2} = m\sum_{j=2}^{\infty} a_j\Delta x^{-j}$ according which the mass is also an implicit function of $\Delta x$, $\Delta\dot{x}$, $\hbar$ and $n$; the lower summation index is due to the intuitive fact that $\Delta\dot{x}$ cannot be function of or proportional to $\Delta x$ otherwise it would diverge for $\Delta x \to \infty$, hence the power series development of the quantity at left hand side must start from $\Delta x^{-2}$. So, putting as usual the coefficient of the first term of the series $a_2 = k_G$, eqs. (4,1) yields $F = -k_G m\Delta x^{-2} + ma_3\Delta x^{-3} + \cdots$. Three remarks on this result are interesting: (i) the first term is nothing else but the Newton gravity field, where now the same $m$ plays also the expected role of gravitational mass generating a radial force that vanishes with $x^{-2}$ law if expressed through the local radial distance $x$ from $m$; (ii) $F$ is in general additive at the first order only, as it is evident considering the sum of $\Delta\dot{x}_1$ due to $F_1$ related to $m_1$ plus an analogous $\Delta\dot{x}_2$ due to $F_2$ in the presence of another mass $m_2$; (iii) gravitational mass generating $F$ and inertial mass defined by $\dot{p}_o$ coincide because in fact $m$ is anyway that uniquely defined in eqs. (4,1). By consequence of (ii) force and acceleration are co-aligned at the first order only. The proportionality factor $k_G$ has physical dimensions $\mathrm{l}^3\mathrm{t}^{-2}$; multiplying and dividing the first term at right hand side by a unit mass $m^u$ and noting that $m^u m$ can be equivalently rewritten as $m'm''$ because $m$ is arbitrary like $m'$ and $m''$, the physical dimensions of $k_G$ turn into $\mathrm{l}^3\mathrm{t}^{-2}\mathrm{m}^{-1}$ while

$$F = -Gm'm''\Delta x^{-2} + m'm''a_3\Delta x^{-3} + \cdots. \qquad (4,2)$$

In conclusion eqs. (2,1) allow to infer as corollaries the two basic statements of general relativity, the arising of inertial forces in accelerated systems and the equivalence principle.

This result legitimates the attempt to extend the approach hitherto outlined to the general relativity, but requires introducing a further remark that concerns the concept of covariance; this concept has to do with the fact that eqs. (4,1) introduce in fact two forces $F_x$ and $F_o$ in inertial, $R$, and non-inertial, $R_o$, reference systems. This early idea introduced by Einstein first in the special relativity and then extended also to the general relativity, aimed to exclude privileged reference systems by postulating the equivalence principle and replacing the concept of gravity force with that of space-time curved by the presence of the mass; Gaussian curvilinear coordinates and tensor calculus are thus necessary to describe the local behavior of a body in a gravity field. This choice allowed on the one side to explain the gedankenexperiment of light beam bending within an accelerated room and on the other side to formulate a covariant theory of universal gravitation through space-time Gaussian coordinates.

Yet the covariancy requires a mathematical formalism that generates conflict with the probabilistic basis of the quantum mechanics: the local metric of the space-time is indeed deterministic, obviously the gravity field results physically different from the quantum fields. It makes really difficult to merge such a way of describing the gravitation with the concepts of non-locality and non-reality that characterize the quantum world. In the present model the concept of force appears instead explicitly: without any "ad hoc" hypothesis the Newton law is obtained as approximate limit case, whereas the transformation from an inertial reference system $R$ to a non-inertial reference system $R_o$ correctly describes the arising of an inertial force.

Hence the present theoretical model surely differs in principle from the special and general relativity; yet, being derived from eqs. (2,1), it is consistent with quantum mechanics as concerns the three key requirements of quantization, non-reality, non-locality. Also, the previous discussion exploits a mathematical formalism that despite its extreme simplicity efficiently bypasses in the cases examined the deterministic tensor formalism of special relativity. In the next sub-section 4.1 attention will be paid to the concept of covariancy, not yet explicitly taken into consideration when introducing the special relativity and apparently skipped so far. Actually this happened because, as shown below, the concept of covariance is already inherent "per se" in the concept of uncertainty once having postulated the complete arbitrariness of size and boundary coordinates of the delocalization ranges.

Let us conclude this introductory discussion rewriting the eqs. (4,1) as $\Delta\dot{p}_x = F = \mu\Delta\ddot{x}$, where

$$\mu = -n\hbar\frac{\Delta\dot{x}}{\Delta x^2\Delta\ddot{x}}$$

has of course physical dimensions of mass; indeed $\Delta\dot{p}_x$ ensures that effectively $\mu$ must somehow be related to the mass of a particle despite it is defined as a function of space delocalization range and its proper time derivatives only.

It is worth noticing that in eq (3,2) the mass was defined regarding the particle as a delocalized corpuscle confined within $\Delta x$, here the quantum of uncertainty $\hbar$ introduces the mass $\mu$ uniquely through its physical dimension. Also note that $\mu/\hbar$ has dimension of a reciprocal diffusion coefficient $D$, so the differential equation $\Delta\dot{x}/(\Delta x^2\Delta\ddot{x}) = \mp(Dn)^{-1}$ admits the solution $\Delta x = (L(\xi) + 1)\sqrt{D\tau_o}$, where $L$ is the Lambert function and $\xi = \pm n\exp(\mp n\Delta t/\tau_o)$; the double sign is due to that possibly owned by $\mu$, the integration constants are $-t_o$ defining $\Delta t = t - t_o$ and $\tau_o$. In conclusion we obtain in the same $R$ of eqs. (4,1)

$$F = \pm n^2\frac{\hbar/\tau_o}{\sqrt{D\tau_o}}\frac{L(\xi)}{(L(\xi)+1)^3}, \quad \frac{\Delta x}{\Delta x_D} = L(\xi) + 1,$$
$$\mu = \pm\hbar/D, \quad \xi = \pm n\exp(\mp n\Delta t/\tau_o), \tag{4,3}$$
$$\Delta x_D = \sqrt{D\tau_o}.$$

Note that the ratio $\Delta\dot{x}/\Delta\ddot{x} = \mp(L(\xi)+1)^2\tau_o/n$ inferred from the given solution never diverges for $n > 0$; moreover $\Delta x$ defined by this solution is related to the well known FLRW parameter $q = -\ddot{a}a/\dot{a}^2$, where $a$ is the scale factor of the universe. Replacing this latter with $\Delta x$ thanks to the arbitrariness of $\Delta x_D$ and $\Delta x$ itself, one finds that $q = \mp L(\xi)^{-1}$.

The importance of eqs. (4,3) rests on the fact that $\Delta x = \Delta x_D$ for $n = 0$ whereas instead, selecting the lower sign, $\Delta x < \Delta x_D$ for any $n > 0$; the reason of it will be clear in the next section 4.3 dealing with the space-time curvature.

It is worth remarking here the fundamental importance of $n$: (i) in [9] its integer character was proven decisive to discriminate between reality/locality and non-reality/non-locality of the classical and quantum worlds; (ii) previously small or large values of $n$ were found crucial to describe relativistic or non-relativistic behavior; (iii) here the values $n = 0$ and $n > 0$ appear decisive to discriminate between an unphysical world without eigenvalues and a physical world as we know it. This last point will be further remarked in the next subsection 4.2.

Eventually $\mu$ deserves a final comment: $\mu$ is a mass defined within $\Delta x$ uniquely because of its $\Delta\dot{x}$ and $\Delta\ddot{x}$; its sign can be in principle positive or negative depending on that of the former or the latter.

Relate $\Delta x$ to the size of our universe, which is still expanding so that $\Delta\dot{x} \neq 0$; also, since there is no reason to exclude that the dynamics of the whole universe corresponds to $\Delta\ddot{x} \neq 0$ too, assume in general an expansion rate not necessarily constant.

It follows for instance $\mu < 0$ if the universe expands at increasing rate, i.e. with $\Delta\dot{x} > 0$ and $\Delta\ddot{x} > 0$. Eqs. (4,3) show that a mass is related to non-vanishing $\Delta x$ and $\Delta\dot{x}$, $\Delta\ddot{x}$. This result appears in fact sensible recalling the dual corpuscle/wave behavior of quantum particles, i.e. imagining the particle as a wave propagating throughout the universe.

It is known that a string of fixed length $L$ vibrates with two nodes $L$ apart, thus with fundamental frequency $\nu_o = v/2L$ and harmonics $\nu_n = n\nu_o = nv/2L$; the propagation velocity of the wave is $v = \nu_n\lambda_n = \sqrt{T/\sigma}$, being $T$ and $\sigma$ the tension and linear density of the string. If $L$ changes as a function of time while the string is vibrating and the wave propagating, then $\nu_n$ and $\lambda_n$ become themselves functions of time.

Let the length change occur during a time $\delta t$; it is trivial to find $\delta\nu_n/\nu_n = (\dot{v}/v - \dot{L}/L)\delta t$, i.e. the frequency change involves $L$, $\dot{L}$ and $\dot{v}$. Put now $L$ equal to the diameter of the universe at a given time, i.e. identify it with $\Delta x$; then propagation rate and frequency of the particle wave clearly change in an expanding universe together with its dynamic delocalization extent.

This therefore means changing the energy $\hbar\delta\nu_n$ of the particle wave, which in turn corresponds to a mass change $\delta m = \hbar\delta\nu_n c^{-2}$. All this agrees with the definition $\mu = \mu(\Delta x, \Delta\dot{x}, \Delta\ddot{x})$ and supports the analogy with the vibrating string. If so the mass $\mu$ results related itself to the big-bang energy, early responsible of the expansion. Once again is the uncertainty the key to highlight the origin of $\mu$: likewise as the time change of $\Delta x$ entails the rising of a force, see eqs. (4,1), correspondingly the time change of the size of the universe changes the delocalization extent of all matter in it contained and thus its internal energy as well.

Two questions arise at this point: has $\mu$ so defined something to do with the supposed "dark mass"? If this latter is reasonably due to the dynamics of our universe and if the kind of this dynamics determines itself both space-time curvature and sign of $\pm\mu$, has this sign to do with the fact that our universe is preferentially made of matter rather than of antimatter? Work is in advanced progress to investigate these points, a few preliminary hints are sketched below.

### 4.1 Uncertainty and covariancy

In general the laws of classical mechanics are not covariant by transformation from inertial to non-inertial reference systems. Their form depends on the arbitrary choice of the reference system describing the time evolution of local coordinates, velocities and accelerations; this choice is subjectively decided for instance to simplify the formulation of the specific problem of interest.

A typical example is that of a tethered mass $m$ rotating frictionless around an arbitrary axis: no force is active in $R$ where the mass rotates, whereas in $R_o$ solidal with the mass is active the centrifugal force; also, if the constrain restraining the mass to the rotation axis fails, the motion of the mass becomes rectilinear and uniform in $R$ but curved in $R_o$, where centrifugal and Coriolis forces also appear. Let in general the non-covariancy be due to a local acceleration $a_R$ in $R$, to which corresponds a combination $a_{R_o}$ of different accelerations in $R_o$. This dissimilarity, leading to fictitious forces appearing in $R_o$ only, suggested to Einstein the need of a covariant theory of gravitation. Just in this respect however the theoretical frame of the present model needs some comments.

First, the local coordinates are conceptually disregarded since the beginning and systematically eliminated according to the positions (2,2), whence the required non-locality and non-reality of the present model; accordingly the functions of coordinates turn into functions of arbitrary ranges, i.e. in 2D $a_R(x,t) \rightarrow a_R(\Delta x, \Delta\varepsilon, \Delta p, \Delta t, n)$, whereas the same holds for $a_{R_o}$. So the classical $x$-components of $a_R$ and $a_{R_o}$ transform anyway into different combinations of the same ranges $\Delta x, \Delta\varepsilon, \Delta p, \Delta t$; the only information is that the local $a_R$ and $a_{R_o}$ become random values within ranges $\Delta a_R = a_R^{(2)} - a_R^{(1)}$ and $\Delta a_{R_o} = a_{R_o}^{(2)} - a_{R_o}^{(1)}$. Yet being these range sizes arbitrary and unpredictable by definition, maybe even equal, is still physically significant now the formal difference between $a_R$ and $a_{R_o}$?

Second, eqs. (4,1) introduce explicitly a force component $F$ via $\Delta \dot{p}_x$ consequence of $\Delta \dot{x} \neq 0$; still appears also in the present model the link between force and deformation of the space-time, hitherto intended however as expansion or contraction of a 2D space-time uncertainty range.

Third, the positions (2,2) discriminate non-inertial, $R_o$, and inertial, $R$, reference systems; from the arbitrariness of $x_o$ and $p_o$ follows that of $\dot{x}_o$ and $\dot{p}_o$ as well. For instance the previous discussion on the 2D eqs. (4,1) leads directly to Einstein's gedankenexperiment of the accelerated box; in the present model the expected equivalence between gravity field in an inertial reference system, $F_x$, and inertial force in accelerated frames, $F'_o$, is indeed obtained simply considering the time dependence of both boundary coordinates of $\Delta x$; without specifying anything, this also entails the equivalence of gravitational and inertial mass. Being all space-time ranges arbitrary, the equivalence principle previously inferred is extensible to any kind of acceleration through a more general, but conceptually identical, 4D transformation from any $R$ to any other $R_o$; indeed defining appropriately $x_{oj}$ and their time derivatives $\dot{x}_{oj}$ and $\ddot{x}_{oj}$ times $m$, with $j = 1,2,3$, one could describe in principle also the inertial forces of the example quoted above through the respective $p_j$, $p_{oj}$ and $\dot{p}_j$, $\dot{p}_{oj}$.

The key point of the present discussion is just here: the arbitrariness of both $x_j$ and $x_{oj}$ generalizes the chances of accounting in principle for any $a_R$ and any $a_{R_o}$. A typical approach of classical physics consists of two steps: to introduce first an appropriate $R$ according which are defined the local coordinates and to examine next the same problem in another $R_o$ via a suitable transformation of these coordinates, whence the necessity of the covariancy. The intuitive considerations just carried out suggest instead that the classical concept of coordinate transformation fails together with that of local coordinates themselves. Imagine an observer able to perceive a range of values only, without definable boundaries and identifiable coordinates amidst; when possibly changing reference system, he could think to the transformation of the whole range only. This is exactly what has been obtained from eqs. (4,1) through the arbitrary time dependence of both $x$ and $x_o$: the classical physics compels deciding either $R$ or $R_o$, the quantum uncertainty requires inherently both of them via the two boundary coordinates of space-time ranges. The ambiguity of forces appearing in either of them only becomes in fact completeness of information, paradoxically just thanks to the uncertainty: the classical freedom of deciding "a priori" either kind of reference system, inertial or not, is replaced by the necessary concurrency of both of them simply because each couple of local dynamical variables is replaced by a couple of ranges.

As shown in the 2D eqs. (4,1), in the present model $R$-like or $R_o$-like reference systems are not alternative options but complementary features in describing any physical system that involves accelerations. Accordingly eqs. (4,1) have necessarily introduced two forces, $F_x$ and $F_o$, related to the two standpoints that entail the equivalence principle as a particular case. After switching the concept of local dynamical variables with that of space-time uncertainty, the physical information turns in general into two coexisting perspectives contextually inferred; inertial and non-inertial forces are no longer two unlike or fictitious images of a unique law of nature merely due to different formulations in $R$ or $R_o$, but, since each one of them requires the other, they generalize the equivalence principle itself. Just this intrinsic link surrogates here the concept of covariancy in eliminating a priori the status of privileged reference system. On the one hand, the chance of observers sitting on accelerated $x_o$ or $x$ excludes by necessity a unique kind of reference system; on the other hand, avoiding fictitious forces appearing in $R_o$ only testifies the ability of the present approach to incorporate all forces into a unique formulation regardless of their inertial and non-inertial nature.

Instead of bypassing the ambiguity of unlike forces appearing in either reference system only by eliminating the forces, the present model eliminates instead the concept itself of privileged reference system in the most general way possible when describing a physical system, i.e. through the concomitant introduction of both $R$ and $R_o$. The total arbitrariness of both boundary coordinates of the uncertainty ranges on the one side excludes a hierarchical rank of $R$ or $R_o$ in describing the forces of nature, while affirming instead the complementary nature of their unique physical essence; on the other side it makes this conclusion true in general, regardless of whether $x_o$ or $x$ is related to the origin $O$ of $R$ and to the size of $\Delta x$.

### 4.2 Uncertainty and space-time curvature

The concept of curvature is well known in geometry and in physics; it is expressed differently depending on the kind of reference system. In general relativity the space-time curvature radius is given by $\rho = g^{ik}R_{ik}$, being $g^{ik}$ the contravariant metric tensor and $R_{ik}$ the Ricci tensor. As already emphasized, however, the central issue to be considered here is not the mathematical formalism to describe the curvature but the conceptual basis of the theoretical frame hitherto exposed; the key point is again that the positions (2,2) exclude the chance of exploiting analytical formulae to calculate the local curvature of the space-time. So, once having replaced the concept of space-time with that of space-time uncertainty, the way to describe its possible curvature must be accordingly reviewed. Just at this stage, eqs. (2,1) are exploited to plug also the quantum non-locality and non-reality in the conceptual structure of the space-time, i.e. into the general relativity.

In a previous paper [9] these features of the quantum world were introduced emphasizing that the measurement process perturbs the early position and momentum of the observed particle, assumed initially in an unphysical state not yet related to any number of states and thus to any observable eigenvalue. Owing to the impossibility of knowing the initial state of the particle, the early conjugate dynamical variables were assumed to fall within the respective $\Delta x^{\S}$ and $\Delta p_x^{\S}$; the notation emphasizes that before the measurement process these ranges are not yet compliant with eqs. (2,1), i.e. they are unrelated. These ranges, perturbed during the measurement process by interaction with the observer, collapse into the respective $\Delta x$ and $\Delta p_x$ mutually related according to the eqs. (2,1) and thus able to define eigenvalues of physical observables through $n$; this also means that $\Delta x^{\S}$ and $\Delta p_x^{\S}$ were mere space uncertainty ranges, whereas after the measurement process only they turn into the respective $\Delta x$ and $\Delta p_x$ that take by virtue of eqs. (2,1) the physical meaning of space-time uncertainty ranges of position and momentum. The paper [9] has explained the reason and the probabilistic character of such a collapse to smaller sized ranges, thanks to which the measurement process creates itself the number of states: the non-reality follows just from the fact that after the measurement process only, the particle leaves its early unphysical state to attain an allowed physical state characterized by the $n$-th eigenvalue.

This kind of reasoning is now conveyed to describe how and why a particle while passing from an unphysical state to any allowed physical state also curves concurrently the space-time. In this way the basic idea of the general relativity, i.e. the space-time curvature, is conceived itself according the concepts of non-reality and non-locality; the latter also follows once excluding the local coordinates and exploiting the uncertainty ranges of eqs. (2,1) only.

To start the argument, note that the arbitrary boundaries of the range $\Delta x^{\S} = x^{\S} - x_o$ control the actual path traveled by a particle therein delocalized. Let the space reference system be an arbitrary 1D $x$-axis about which nothing is known; information like flat or curled axis is inessential. Thus the following considerations are not constrained by any particular hypothesis on the kind of possible curvature of the early 1D reference system. Consider first the space range $\Delta x^{\S}$ alone; changing by an arbitrary amount $dx^{\S}$ the actual distance of $x^{\S}$ from $x_o$ on the $x$-axis, the size of $\Delta x^{\S}$ changes as well so that $d\Delta x^{\S}/dx^{\S} = 1$, i.e. $d\Delta x^{\S} = dx^{\S}$. This implicitly means that the range $\Delta x^{\S}$ overlaps to, i.e. coincides with, the reference $x$-axis. Thus the delocalization motion of the particle lies by definition between the aforesaid boundary coordinates just on this axis, whatever its actual geometry before the measurement process might be. In principle this reasoning holds for any other uncertainty range corresponding to $\Delta x^{\S}$, e.g. the early local energy of a particle delocalized within $\Delta x^{\S}$ could be a function of its local coordinate along the $x$-axis; however such a local value of energy is inconsequential, being in fact unobservable in lack of $n$ and thus by definition unphysical.

Consider again the aforesaid 1D space range, yet assuming now that a measurement process is being carried out to infer physical information about the particle; as a consequence of the perturbation induced by the observer, the actual correlation of $\Delta x = x - x_o$ with its conjugate range $\Delta p_x = p_x - p_o$ of allowed momenta introduces $n$ too; now, by virtue of eqs. (2,1), these ranges take the physical meaning of space-time uncertainties and concur to define allowed eigenvalues according to the concept of quantum non-reality. Although $\Delta x$ is still expressed by two arbitrary coordinates on the $x$-axis, it is no longer defined by these latter only; rather $\Delta x$ is defined taking into account also its correlation with $\Delta p_x$ through $n$. In other words eqs. (2,1) compel regarding the change of $x$, whatever it might be, related to that of $\Delta p_x$; this does not contradict the concept of arbitrariness of the ranges so far assumed, as $x$ remains in fact arbitrary like $\Delta p_x$ itself and unknown like the function $x(\Delta p_x)$ correlating them. Yet, when calculating $d\Delta x/dx$ with the condition $\Delta x\Delta p_x = n\hbar$, we obtain in general $d\Delta x/dx = -(n\hbar)^{-1}\Delta x^2 d\Delta p_x/dx \neq 1$.

To summarize, $\Delta x^\S$ and $\Delta x$ have not only different sizes but also different physical meaning, i.e. the former is mere precursor of the latter: before the measurement process $\Delta x^\S$ overlapped to the $x$-axis and had mere space character, the early path length of the particle lay on the reference axis, i.e. $d\Delta x^\S = dx^\S$; after the measurement process $\Delta x^\S$ shrinks into the new $\Delta x$ such that in general $d\Delta x \neq dx$, thus no longer coincident with the $x$-axis and with space-time character. In this way the measurement process triggers the space-time uncertainty, the space-time curvature and the allowed eigenvalues as well.

Let us visualize for clarity why the transition from space to space-time also entails curved Gaussian coordinates as a consequence of the interaction of the particle with the observer. If $\Delta x^\S$ shrinks to $\Delta x$, then the early boundary coordinates of the former must somehow approach each other to fit the smaller size of the latter; thus the measurement driven contraction pushes for instance $x^\S$ towards a new $x$ closer to $x_o$ along the reference axis previously coinciding with the space range $\Delta x^\S$ and its possible $dx^\S$. So, after shrinking, $\Delta x^\S$ turns into a new bowed space-time range, $\Delta x$, forcedly decoupled from the reference $x$-axis because of its acquired curvature, whence $dx \neq dx^\S$ as well. If length of the $x$-axis and size of the uncertainty range physically allowed to delocalize the particle do no longer coincide, the particle that moves between $x_o$ and $x$ follows actually a bowed path reproducing the new curvature of $\Delta x$, no longer that possibly owned by the 1D reference system itself, whence the curvature of the 2D space-time uncertainty range.

This is possible because nothing is known about the actual motion of the particle between the boundary coordinates $x_o$ and $x$ of the reference $x$-axis; moreover it is also possible to say that the new curvature is due to the presence of a mass in $\Delta x^\S$, as in lack of a particle to be observed the reasoning on the measurement process would be itself a non-sense.

The last remark suggests correctly that the space-time is actually flat in the absence of matter, as expected from the original Einstein hypothesis, so is seemingly tricky the previous specification that even the early $\Delta x^\S$ could even owe a possible curvature coincident with that of the $x$-reference axis; this specification, although redundant, was deliberately introduced to reaffirm the impossibility and uselessness of hypotheses on the uncertainty ranges and to avoid confusion between arbitrariness of the uncertainty ranges and Einstein's hypothesis.

Eventually, the probabilistic character of the shrinking of delocalization range, emphasized in [9], guarantees the probabilistic nature of the origin of space-time and its curvature. Indeed all above is strictly related to the time uncertainty: a time range $\Delta t$ is inevitably necessary to carry out the measurement process during which $\Delta x^\S$ and $\Delta p_x^\S$ collapse into $\Delta x$ and $\Delta p_x$.

As found in the previous section, the correlation of the range deformation with the time involves change of momentum of the particle within $\Delta p_x$, i.e. the rising of a force component as previously explained. This reasoning therefore collects together four concepts: (i) introduces the space-time as a consequence of the measurement process starting from an unphysical state of the particle in a mere space range and in an unrelated momentum range, both not compliant separately with observable eigenvalues; (ii) introduces the non-reality into the space-time curvature, triggered by the measurement process; (iii) links a force field to this curvature by consequence of the measurement process; (iv) introduces the uncertainty into the concepts of flat space and curved space-time: the former is replaced by the idea of an early space uncertainty range where is delocalized the particle coincident with the coordinate axis, whatever its actual geometry might be; the latter is replaced by the idea of early geometry modified by the additional curvature acquired by the new $\Delta x$ with respect to that possibly owned by the $x$-axis during their decoupling. Of course just this additional curvature triggered by the measurement process on the particle present in $\Delta x^\S$ is anyway that experimentally measurable.

In conclusion, the measurement process not only generates the quantum eigenvalues of the particle, and thus its observable properties described by their number of allowed states, but also introduces the space-time inherent eqs. (2,1) concurrently with new size and curvature with respect to the precursor space delocalization range. Hence the particle is effectively confined between $x_o$ and $x$ during the time range $\Delta t$; yet, in the 2D feature of the present discussion, it moves outside the reference axis. Actually these conclusions have been already inferred in eqs. (4,3); it is enough to identify $\Delta x^\S$ with the previous $\Delta x_D$ for $n = 0$ to find all concepts so far described.

Note that the existence of a curved space-time was not explicitly mentioned in section 3, in particular when calculating the orbital and spin angular momenta or hydrogenlike energy in subsection 3.3, simply because it was unnecessary and inconsequential: the eigenvalues do not depend on the properties of the uncertainty ranges, e.g. on their sizes and possible curvature, nor on the random values of local dynamical variables therein defined. To evidence either chance of flat or curved space-time uncertainty, the next sub-section 4.3.2 describes the simulation of a specific physical experiment, the light beam bending in the presence of a gravitational mass, whose outcome effectively depends on the kind of path followed by the particle.

This "operative" aspect of the model is indeed legitimate now; after having introduced the basic requirements of special and general relativity and a possible explanation of the space-time curvature, we are ready to check whether or not some significant outcomes of general relativity can be effectively obtained in the conceptual frame of eqs. (2,1) through the positions (2,2) only. Once again, the essential requirement to merge relativity and quantum mechanics is to regard the deterministic intervals of the former as the quantum uncertainty ranges of the latter.

### 4.3 Some outcomes of general relativity

Before proceeding on, it is useful a preliminary remark. Despite the conceptual consistency of eqs. (2,1) with the special relativity, extending an analogous approach to the general relativity seems apparently difficult.

Consider for instance the time dilation and the red shift in the presence of a stationary gravitational potential $\varphi$. As it is known, the general relativity achieves the former result putting $dx^1 = dx^2 = dx^3 = 0$ in the interval $-ds^2 = g_{ik}dx^i dx^k$; calculating the proper time in a given point of space as $\tau = c^{-1}\int\sqrt{-g_{00}}dx^0$, the integration yields $\tau = c^{-1}x^0\sqrt{1+2\varphi/c^2}$, i.e. $\tau = c^{-1}x^0(1+\varphi/c^2)$.

In an analogous way is calculated the red shift $\Delta\omega = c^{-2}\omega\Delta\varphi$ between two different points of space where exists a gap $\Delta\varphi$ of gravitational potential $\varphi$. Are the ranges of eqs. (2,1) alone suitable and enough to find similar results once having discarded the local conjugate variables?

Appears encouraging in this respect the chance of having obtained as corollaries the fundamental statements of special and general relativity. Moreover is also encouraging the fact that some qualitative hints highlight reasonable consequences of eqs. (2,1).

Put $m' = \hbar\omega/c^2$ to describe a system formed by a photon in the gravity field of the mass $m$; thus $\Delta\dot{p}_x = F$ of eq (4,1) is now specified as the momentum change of the photon because of the force component $F$ due to $m$ acting on $m'$. Since the photon moves in the vacuum at constant velocity $c$ there are two possibilities in this respect: the photon changes its wavelength or its propagation direction.

These chances correspond to two relevant outcomes of general relativity, i.e. the red shift and the light beam bending in the presence of a gravity field; the former occurs when the initial propagation direction of the photon coincides with the $x$-axis along which is defined the force component $\Delta\dot{p}_x$, i.e. radial displacement, the latter when the photon propagates along any different direction. The bending effect is of course closely related to the previous considerations about the actual curvature of the space-time uncertainty range that makes observable the path of the photon; this means that in fact the deflection of the light beam replicates the actual profile of $\Delta x$ with respect to the $x$-axis.

Eventually, also the perihelion precession of orbiting bodies is to be expected because of non-Newtonian terms in eq (4,2); it is known indeed that the mere gravitational potential of Newton law allows closed trajectories only [12].

From a qualitative point of view, therefore, it seems that the results of general relativity should be accessible also in the frame of the present theoretical approach. It is necessary however to explain in detail how the way of reasoning early introduced by Einstein is replaced here to extend the previous results of special relativity. The following subsections aim to show how to discuss the curvature of the space-time uncertainty range and then how to describe time dilation, red shift and light beam bending exploiting uniquely the uncertainty ranges of eqs. (2,1) only, exactly as done at the beginning of section 3.

#### 4.3.1 The time dilation and the red shift

Infer from eqs. (2,1) $\Delta x\Delta p_x/\Delta t = n\hbar/\Delta t$, which also reads $m\Delta x\Delta v_x/\Delta t = n\hbar/\Delta t$. Holds also here the remark introduced about eqs. (4,1), i.e. the particular boundary values of $p_o$ and $p_x$ determining the size of the momentum range $\Delta p_x = p_x - p_o$ are arbitrary, not specifiable in principle and indeed never specified; therefore, since neither $p_o$ nor $p_x$ need being calculated, the actual expression of local momentum is here inessential. So, merely exploiting the physical dimensions of momentum, it is possible to replace $\Delta p_x$ with $m\Delta v_x$ and write $m\Delta v_x\Delta x/\Delta t = n\hbar/\Delta t$, whatever $\Delta v_x$ and $m$ might in fact be. Hence, the energy at right hand side can be defined as follows

$$m\varphi_x = -\frac{n\hbar}{\Delta t}, \qquad \varphi_x = -\Delta x\frac{\Delta v_x}{\Delta t}, \qquad \varphi_x < 0. \tag{4,4}$$

Being the range sizes positive by definition, $\varphi_x$ has been intentionally introduced in the first equation with the negative sign in order that $m\varphi_x = -\Delta\varepsilon$ correspond to an attractive force component $F = -\Delta\varepsilon/\Delta x$ of the same kind of the Newton force, in agreement with the conceptual frame of relativity. Also, $\varphi_x$ does not require specifying any velocity because for the following considerations is significant its definition as a function of $\Delta v_x$ only. This result can be handled in two ways.

In the first way, the first eq. (4,4) is rewritten as follows

$$-\frac{\hbar}{\Delta t} = \varepsilon\frac{\varphi_x}{c^2}, \qquad \varepsilon = (m/n)c^2, \tag{4,5}$$

in which case one finds

$$\frac{\Delta t - t_o}{\Delta t} = 1 + \frac{\varphi_x}{c^2}, \qquad \frac{\hbar}{\varepsilon} = t_o,$$

$$\frac{m\varphi_x}{\Delta x} = -m\frac{\Delta v_x}{\Delta t} = -F_N. \tag{4,6}$$

Note that $t_o$ is a proper time of the particle, because it is defined through the energy of this latter. In this case the number $n$ is unessential and could have been omitted: being the mass $m$ arbitrary, $m/n$ is a new mass arbitrary as well. The third result defines $\varphi_x$ as a function of the expected Newton force component $F_N$; hence $\varphi_x$ corresponds classically to a gravitational potential. The first equation is interesting: it correlates through $\varphi_x$ the time ranges $\Delta t' = \Delta t - t_o$ and $\Delta t$. Note that if $\varphi_x \to 0$ then $\Delta t \to \infty$ according to eqs. (4,4) or (4,5), i.e. $\Delta t' \to \Delta t$; hence the gravitational potential $\varphi_x$ provides a relativistic correction of $\Delta t$, which indeed decreases to $\Delta t'$ for $\varphi_x \neq 0$. Eq. (4,6) is thus just the known expression $\tau = (x_0/c)(1+\varphi_x/c^2)$ previously reported once replacing $\tau/(c^{-1}x_0)$ with $\Delta t'/\Delta t$; indeed in the present approach the local quantities are disregarded and replaced by the corresponding ranges of values. The first eq (4,6) shows that time slowing down $\Delta t - t_o$ occurs in the presence of a gravitational potential with respect to $\Delta t$ pertinent to $\varphi_x = 0$.

The second way to handle eqs. (4,4) consists of considering two different values of $\varphi_x$ at its right hand side and a particle that climbs the radial gap corresponding to the respective values of gravitational potential with respect to the origin of an arbitrary reference system; moreover, being $\varepsilon$ constant by definition because $t_o$ is fixed, the proper times of the particle $t_1$ and $t_2$ define the corresponding time ranges $\Delta t_1$ and $\Delta t_2$ necessary for the particle to reach the given radial distances. So eqs. (4,5) yield with obvious meaning of symbols

$$-\frac{\hbar/\varepsilon}{\Delta t^{(1)}} = \frac{\varphi_x^{(1)}}{c^2} \qquad -\frac{\hbar/\varepsilon}{\Delta t^{(2)}} = \frac{\varphi_x^{(2)}}{c^2}.$$

Hence, putting $\omega = \Delta t^{-1}$, one finds

$$\frac{\omega_1 - \omega_2}{\omega_o} = \frac{\varphi_x^{(2)} - \varphi_x^{(1)}}{c^2}, \qquad \omega_o = \frac{\varepsilon}{\hbar}. \tag{4,7}$$

Here $\omega_o$ is the proper frequency of the free photon with respect to which are calculated $\omega_1$ and $\omega_2$ at the respective radial distances. This expression yields the frequency change between two radial distances as a function of $\omega_o$

$$\Delta\omega = \frac{\Delta\varphi_x}{c^2}\omega_o.$$

Since $\varphi_x$ is negative, the sign of $\Delta\omega$ is opposite to that of $\Delta\varphi_x$: if $\varphi_x^{(2)}$ is stronger than $\varphi_x^{(1)}$, then $\varphi_x^{(2)} - \varphi_x^{(1)} < 0$, which means that $\omega_2 > \omega_1$. One finds the well known expression of the red shift occurring for decreasing values of gravitational potential. We have inferred two famous result of general relativity through uncertainty ranges only. Now we can effectively regard these results as outcomes of quantum relativity.

### 4.3.2 The light beam bending

Rewrite eq (4,2) as $F_N\Delta x/(\hbar\omega/c^2) = -Gm/\Delta x$; here $F_N$ is due to the mass $m$ acting along the $x$ direction on a photon having frequency $\omega$ and traveling along an arbitrary direction; the notation emphasizes that the photon energy $\hbar\omega/c^2$ replaces the mass of a particle in the gravity field of $m$. The distance between photon and $m$ is of course included within $\Delta x$. Introduce with the help of eq (4,4) the gravitational potential $\varphi_x = -F_N\Delta x/m$, so that $\varphi_x/c^2 = Gm/(c^2\Delta x)$. Now it is possible to define the beam deflection through $\varphi_x$, according to the idea that the beam bending is due just to the gravitational potential; we already know why this effect is to be in fact expected. Of course, having discarded the local coordinates, the reasoning of Einstein cannot be followed here; yet since $\delta\phi = \delta\phi(\varphi_x)$, with notation that emphasizes the dependence of the bending angle $\delta\phi$ of the photon upon the field $\varphi_x$, it is certainly possible to express the former as series development of the latter, i.e. $\delta\phi = \alpha + \beta(\varphi_x/c^2) + \gamma(\varphi_x/c^2)^2 + \cdots$; $\alpha$, $\beta$ and $\gamma$ are coefficients to be determined. Clearly $\alpha = 0$ because $\delta\phi = 0$ for $m = 0$, i.e. there is no bending effect; so

$$\delta\phi \approx \frac{Gm\beta}{c^2\Delta x}, \qquad \frac{Gm}{c^2\Delta x} \approx \frac{-\beta + \sqrt{\beta^2 + 4\gamma\delta\phi}}{2\gamma}. \tag{4,8}$$

The former expression is simpler but more approximate than the latter, because it account for one term of the series development of $\delta\phi(\varphi_x)$ only; the latter calculates instead $\varphi_x$ as a function of $\delta\phi$ at the second order approximation for reasons that will appear below. Consider first the former expression and note that even in lack of local coordinates the deflection can be expressed as the angle between the tangents to the actual photon path at two arbitrary ordinates $y_-$ and $y_+$ along its way: i.e., whatever the path of the photon might be, we can figure $m$ somewhere on the $x$-axis and the photon coming from $-\infty$, crossing somewhere the $x$ axis at any distance within $\Delta x$ from $m$ and then continuing a bent trajectory towards $+\infty$. Let the abscissas of the arbitrary points $y_-$ and $y_+$ on the $x$-axis be at distances $\Delta x_-$ and $\Delta x_+$ from $m$; the tangents to these points cross somewhere and define thus an angle $\delta\phi'$. The sought total deflection $\delta\phi$ of the photon corresponds thus to the asymptotic tangents for $y_-$ and $y_+$ tending to $-\infty$ and $\infty$. Note now that the same reasoning holds also for a reversed path, i.e. for the photon coming from infinity and traveling towards minus infinity; the intrinsic uncertainty affecting these indistinguishable and identically allowed chances suggests therefore a boundary condition to calculate the change of photon momentum $h/\lambda$ during its gravitational interaction with the mass. The impossibility of distinguishing either chance requires defining the total momentum range of the photon as $\Delta p = h/\lambda - (-h/\lambda) = 2h/\lambda$, i.e. $\Delta p = (2/c)\hbar\omega$. Since the momentum change depends on $c/2$, and so also the interaction strength $\Delta p/\Delta t$ corresponding to $F_N$, it is reasonable to assume that even $\delta\phi$ should depend on $c/2$; so putting $\beta = 4$ in the former expression of $\delta\phi$ and noting that the maximum deflection angle calculated for $y_- \to -\infty$ and $y_+ \to +\infty$ corresponds to the minimum distance range $\Delta x$, one finds the well known result

$$\delta\phi \approx \frac{4Gm}{c^2\Delta x_{\min}}.$$

The numerical factor 4 appears thus to be the fingerprint of the quantum uncertainty, whereas the minimum approach distance of the Einstein formula is of course replaced here by its corresponding uncertainty range $\Delta x_{\min}$. It is also interesting to consider the second equation (4,8), which can be identically rewritten as follows putting $\gamma = \gamma'\beta$ and again $\beta = 4$ to be consistent with the previous result as a particular case; so

$$\rho = \frac{\sqrt{1+\gamma'\delta\phi} - 1}{\gamma'}, \qquad \rho = \frac{r_{Schw}}{\Delta x_{\min}}, \qquad r_{Schw} = \frac{2Gm}{c^2},$$

with the necessary convergence condition of the series that reads $\left|\gamma'\varphi_x/c^2\right| < 1$ and requires

$$\frac{\sqrt{1+\gamma'\delta\phi}-1}{2} < 1.$$

This condition requires $-\delta\phi^{-1} \leq \gamma' < 8\delta\phi^{-1}$, and therefore $r_{Schw}\delta\phi^{-1} \leq \Delta x_{\min} < 4r_{Schw}\delta\phi^{-1}$. Replace in this result $\delta\phi = \pi$ and consider what happens when a photon approaches $m$ at distances $r_{bh}$ between $\pi^{-1}r_{Schw} < r_{bh} < 4\pi^{-1}r_{Schw}$: (i) the photon arrives from $-\infty$ and makes half a turn around $m$; (ii) after this one half turn it reaches a position diametrically opposite to that of the previous step; (iii) at this point the photon is still in the situation of the step (i), i.e. regardless of its provenience it can make a further half a turn, and so on. In other words, once arriving at distances of the order of $2Gm/c^2$ from $m$ the photon starts orbiting without possibility of escaping; in this situation $m$ behaves as a black body. Here the event horizon turns actually into a range of event horizons, i.e. into a shell surrounding $m$ about $\sim 3\pi^{-1}r_{Schw}$ thick where the gravitational trapping is allowed to occur; this result could be reasonably expected because no particle, even the photon, can be exactly localized at some deterministic distance from an assigned point of space-time, i.e. the event horizon is replaced by a range of event horizons. Note however that the reasoning can be repeated also imposing $\delta\phi = 2\pi$ and, more in general, $\delta\phi = 2j\pi$ where $j$ describe the number of turns of the photon around $m$. In principle the reasoning is the same as before, i.e. after $j$ revolutions required by $\delta\phi$ the photon is allowed to continue again further tours; yet now trivial calculations yield $(j\pi)^{-1}r_{Schw} < r_{bh} < 4(j\pi)^{-1}r_{Schw}$. At increasing $j$ the shell allowing the turns of the photon becomes thinner and thinner while becoming closer and closer to $m$. As concerns the ideal extrapolation of this result to approach distances $r_{bh} < \pi^{-1}r_{Schw}$ one can guess for $j \to \infty$ the chance of photons to spiral down and asymptotically fall directly on $m$ without a stable orbiting behavior.

#### 4.3.3 The Kepler problem and the gravitational waves

The problem of perihelion precession of planets is too long to be repeated here even in abbreviated form. It has been fully concerned in a paper preliminarily submitted as preprint [13]. We only note here how this problem is handled in the frame of the present model. It is known that the precession is not explained in the frame of classical mechanics. If the potential energy has the form $-\alpha/r$ the planet follows a closed trajectory; it is necessary a form of potential energy like $\alpha/r + \delta U$ to describe the perihelion precession. The Newton law entails the former kind of potential energy, but does not justifies the correction term $\delta U$. In our case, however, we have found the Newton law as a particular case of a more general force containing additional terms, eq (4,2); thanks to these latter, therefore, it is reasonable to expect that the additional potential term enables the perihelion precession to be described. Also in this case the formula obtained via quantum uncertainty ranges coincides with the early Einstein formula. The same holds for the problem of the gravitational waves, also concerned together with some cosmological considerations in the quoted preprint. Both results compel regarding once again the intervals of relativity as uncertainty ranges.

#### 4.3.4 Preliminary considerations on eqs. (4,3)

This subsection introduces preliminary order of magnitude estimates on the propagation wave corresponding to the mass $\mu = \hbar/D$; the $\pm$ sign is omitted because the following considerations concern the absolute value of $\mu$ only.

Consider a wave with two nodes at a diametric distance $d_u$ on a sphere simulating the size of universe; the first harmonic has then wavelength $\lambda_u = 2d_u$. Let the propagation rate $v$ of such a wave be so close to $c$, as shown below, that for brevity and computational purposes only the following estimates are carried out replacing directly $v$ with $c$. Guess the quantities that can be inferred from $D$ by means of elementary considerations on its physical dimensions in a reference system $R$ fixed on the center of the whole universe. Calculate $D$ as $\lambda_u$ times $c$, i.e. $D = 2d_uc$, and define $\tau$ as $\sqrt{D\tau} = d_u/2$, i.e. as the time elapsed for $\mu$ to cover the radial distance of the universe; so $\tau$ describes the growth of the universe from a size ideally tending to zero at the instant of the big-bang to the current radius $\sqrt{D\tau}$. Since $\lambda_u = 0$ at $\tau = 0$ and $\lambda_u = 2d_u$ at the current time $\tau$, then $d_u = 8c\tau$ and $D = 16c^2\tau$. Moreover, considering that $G$ times mass corresponds to $D$ times velocity, guess that $m_u = 16c^3\tau/G$ introduces the mass $m_u$ to which correspond the rest energy $\varepsilon_u = 16c^5\tau/G$ and rest energy density $\eta_u = 3c^2/(16\pi G\tau^2)$ calculated in the volume $V_u = 4\pi(d_u/2)^3/3$ of the universe. Also, the frequency $\omega_\mu = \xi c^2/D$ of the $\mu$-wave defines the zero point energy

$$\varepsilon_{zp} = \hbar\omega_\mu/2 = \mu'c^2/2 \qquad \mu' = \xi\mu$$

of oscillation of $\mu$; the proportionality constant $\xi$ will be justified below. At right hand side appears the kinetic energy of the corpuscle corresponding to $\hbar\omega_\mu/2$, in agreement with the mere kinetic character of the zero point energy. Note that with trivial manipulations $D = 16c^2\tau$ reads also in both forms

$$\frac{\hbar^2}{2\mu(d_u/2)^2} = \frac{\hbar}{2\tau} \qquad \lambda_\mu = d_u/2 = \frac{\hbar}{\mu c} \qquad (4,9)$$

The left hand side of the first equation yields $\varepsilon_{zp}$ of the $\mu$-corpuscle, also calculable from $\Delta p_{zp}^2/2\mu$ i.e. $\hbar^2/2\mu\Delta x_{zp}^2$ replacing $\Delta x_{zp}$ with $d_u/2$; this means that the momentum of a free unbounded particle initially equal to an arbitrary value $p_1$ increases to $p_2$ after confinement in a range $\Delta x_{zp}$, whence the conjugate range $\Delta p_{zp} = p_2 - p_1$. Equating this result to $\mu c^2/2$ one finds the second equation, which shows that the Compton length of the $\mu$-particle is the universe radius. Also $\hbar/2\tau$

must describe a zero point energy; this compels introducing the frequency $\omega_u = 1/\tau$ so that it reads $\hbar\omega_u/2$.

Define now the ratio $\sigma_\mu = \mu D/\mathrm{V}_\mu\omega_\mu$ to express the linear density of $\mu$ as a function of its characteristic volume $\mathrm{V}_\mu$ and length $\Delta x_\mu = \mathrm{V}_\mu\omega_\mu/D$: since the squared length inherent $D$ concerns by definition a surface crossed by the particle per unit time, $\Delta x_\mu$ lies along the propagation direction of $\mu$. This way of defining $\sigma_\mu = \mu/\Delta x_\mu$ is thus useful to calculate the propagation velocity of the $\mu$-wave exploiting the analogy with the string under tension $T$; so $v = \sqrt{T/\sigma_\mu}$ yields $T = \hbar c^2/\mathrm{V}_\mu\omega_\mu$, which in fact regards the volume $\mathrm{V}_\mu$ as a physical property of the mass $\mu$. This expression of $T$ appears reasonable recalling that $\mu$ is defined by the ratio $\Delta\dot{x}\Delta\ddot{x}^{-1}\Delta x^{-2}$ of uncertainty ranges, which supports the idea of calculating its mass linear density within the space-time uncertainty range $\Delta x_\mu$ that defines $\sigma_\mu$ through $\mathrm{V}_\mu$. Consider that also the ratio $v^2/G$ has the dimension of mass/length; replacing again $v$ with $c$ we obtain $c^2 = TG/c^2$, i.e. the tension of the string corresponds to a value of $F$ of eqs. (4,3) of the order of the Planck force acting on $\mu$; so, comparing with the previous expression of $T$, one infers $\mathrm{V}_\mu \approx \hbar G/\omega_\mu c^2$, i.e. $\mathrm{V}_\mu \approx \hbar DG/c^4$. Thus $\mathrm{V}_\mu$ has a real physical identity defined by the fundamental constants of nature and specified to the present problem by $\omega_\mu^{-1}$.

Before commenting this point, let us show that the actual propagation velocity of the $\mu$-wave is very close to $c$. Exploit the wave and corpuscle formulae of the momentum of $\mu$ putting $h/\lambda_u = \mu v/\sqrt{1-(v/c)^2}$ i.e. $2\pi\sqrt{1-(v/c)^2} = (v/c)$; then $v \approx 0.99c$ justifies the expressions inferred above, whereas $\varepsilon_\mu = \mu c^2/\sqrt{1-(v/c)^2}$ is about 6.4 times the rest value $\mu c^2$. Call $\xi$ this kinetic correction factor. In principle all expressions where appears explicitly $\mu$ still hold, replacing however this latter with $\mu' = \xi\mu$ as done before; it explains why $\omega_\mu$ has been defined just via $\xi$. This is also true for $\varepsilon'_\mu = \mu' c^2$, for $\varepsilon'_{zp} = \varepsilon_{zp}(\mu')$ and for the effective Compton length $\lambda'_\mu$, which result therefore slightly smaller than $d_u/2$ because it is the Loretz contraction of the proper length $\lambda_\mu$, but not for $\omega_u$, whose value is fixed by $\tau$ and $d_u$. Indeed at this point is intuitive to regard $\tau$ as a time parameter as a function of which are calculated all quantities hitherto introduced.

Before considering this problem let us introduce the particular value of $\tau$ equal to the estimated age of our universe, commonly acknowledged as about $4 \times 10^{17}$s; this yields the following today's time figures:

$$d_u = 9.6 \times 10^{26}\mathrm{m}, \qquad m_u = 2.6 \times 10^{54}\mathrm{kg},$$

$$\omega_u = 2.5 \times 10^{-18}\mathrm{s}^{-1}, \qquad \varepsilon_u = 2.3 \times 10^{71}\mathrm{J},$$

$$\eta_u = 5.0 \times 10^{-10}\mathrm{Jm}^{-3}, \qquad \hbar\omega_u/2 = 1.3 \times 10^{-52}\mathrm{J},$$

and also

$$D = 5.8 \times 10^{35}\mathrm{m}^2\mathrm{s}^{-1}, \qquad \omega_\mu = 9.9 \times 10^{-19}\mathrm{s}^{-1},$$

$$\mu = 1.8 \cdot 10^{-70}\mathrm{kg}, \qquad \mu' = 1.2 \times 10^{-69}\mathrm{kg},$$

$$\varepsilon'_\mu \approx 1.0 \times 10^{-52}\mathrm{J}, \qquad \hbar\omega_\mu/2 = 5.2 \times 10^{-53}\mathrm{J}.$$

It is interesting the fact that the results split into two groups of values: the quantities with the subscript $u$ do not contain explicitly $\mu$ and are in fact unrelated to $D$, $\omega_\mu$ and $\varepsilon_\mu$. Are easily recognized the diameter $d_u$ and the mass $m_u$ of matter in the universe, which support the idea that just the dynamics of the universe, i.e. $\Delta\dot{x}$ and $\Delta\ddot{x}$, concur together with its size, i.e. $\Delta x$, to the mass in it present.

This was indeed the main aim of these estimates. The average rest mass density $m_u/V_u$ is about $5.6 \times 10^{-27}\mathrm{Kg/m}^3$. Is certainly underestimated the actual energy $\varepsilon_u$, here calculated without the kinetic Lorentz factor taking into account the dynamic behavior of $m_u$, i.e. the average velocity of the masses in the universe; $\varepsilon_u$ and thus $\eta_u$ are expected slightly greater than the quoted values. However this correction factor can be neglected for the present purposes because it would be of the order of a few % only at the ordinary speed with which moves the matter. The order of magnitude of the energy density $\eta_u$, of interest here, is close to that expected for the average vacuum energy density $\eta_{vac}$; it suggests $\eta_u = \eta_{vac}$, i.e. the idea that matter and vacuum are a system at or near to the dynamic equilibrium based on creation and annihilation of virtual particles and antiparticles. This way of linking the energy densities of $\mu$ and matter/vacuum emphasizes that the dynamic of the universe, regarded as a whole system, concerns necessarily its total size and life time; this clearly appears in eqs. (4,9) and is not surprising, since $\mu$ is consequence itself of the space-time evolution $\Delta\dot{x}\Delta\ddot{x}^{-1}\Delta x^{-2}$ of the universe.

Note now the large gap between the values of $\mu$ and $m_u$: this is because the former is explicit function of $D$, the latter does not although inferred in the frame of the same reasoning. Despite the different values and analytical form that reveal their different physical nature, a conceptual link is therefore to be expected between them. Let the characteristic volume $\mathrm{V}_\mu$ be such that $\varepsilon'_{zp}/\mathrm{V}_\mu = \eta_{vac} = \eta_u$, which requires $\mathrm{V}_\mu = 8\pi G\tau^2\mu'/3$. This means that the universe evolves keeping the average energy density due to the ordinary matter, $\eta_u$, in equilibrium with that of the vacuum, $\eta_{vac}$, in turn triggered by the zero point energy density of $\mu'$ delocalized in it: in this way both $\eta_{vac}$ and $\eta_u$ result related to the early big-bang energy and subsequent dynamics of the universe described by $\mu$. To verify this idea, get some numbers: $\mathrm{V}_\mu = 8\pi G\tau^2\mu'/3$ results about $1.0 \times 10^{-43}\mathrm{m}^3$, whereas $\mathrm{V}_\mu = \hbar G/\omega_\mu c^2$ yields the reasonably similar value $7.9 \times 10^{-44}\mathrm{m}^3$. Moreover there is a further significant way to calculate $\mathrm{V}_\mu$. Define the volume $V_\mu = \pi(d_u/2)^2\Delta x_\mu$ and rewrite identically $\Delta x_\mu = \hbar G/Dc^2$, having put $T$ just equal to the Planck force; one finds $V_\mu = \pi\hbar G\tau/c^2$ i.e. $V_\mu = 9.8 \times 10^{-44}\mathrm{m}^3$ that agrees with the previous values although it does not depend on $\mu$ and thus on the correction factor $\xi$. In other words, $\xi$ could have been also calculated in order that $\omega$ and $\mu'$ fit this last value of $\mathrm{V}_\mu$; of course the result would agree with the relativistic wave/corpuscle behavior of $\mu$.

These outcomes confirm the consistency of the ways to calculate $V_\mu$ and the physical meaning of $\mu'$, in particular the considerations about $T$. Yet the most intriguing result is that the size of $V_\mu$ also comes from a very large number, the area of a diametric cross section of the universe, times an extremely small number, the thickness $\Delta x_\mu = 8.6 \times 10^{-97}$m used to calculate the linear density $\sigma_\mu$ and thus $T$. Of course any diametric section is indistinguishable from and thus physically unidentifiable with any other section, otherwise should exist some privileged direction in the universe; so the volume $V_\mu$, whatever its geometrical meaning might be, must be regarded as permeating all universe, in agreement with the concept of delocalization required by eqs. (2,1).

Despite $\mu' c^2/2$ is a very small energy, its corresponding energy density accounts in fact for that of the vacuum because of the tiny value of $V_\mu$. Compare this estimate with that of $m_u c^2$ intuitively regarded in the total volume $V_u$ of the universe: so as $V_u$ is the characteristic volume of ordinary matter, likewise $V_\mu$ is the characteristic volume of $\mu$ i.e. a sort of effective physical size of this latter. Since $\mu' > \mu$, the first eq (4,9) includes in $V_\mu$ an excess of zero point energy with respect to that previously calculated with $\mu'$; just for this reason indeed $\hbar\omega_u/2 > \hbar\omega'_\mu/2$. The previous expressions of $\varepsilon'_{zp}$ account for the actual kinetic mass $\mu'$ by replacing the rest mass $\mu$. Yet in the first eq (4,9) this is not possible because $\tau$, once fixed, is consistent with $\mu$ and not with $\mu'$. The simplest idea to explain this discrepancy is that actually $\hbar/2\tau$ accounts for two forms of energy: the zero point energy, which can be nothing else but $\xi\mu c^2/2$ previously inferred, plus an extra quantity

$$\delta\varepsilon = \hbar^2\mu^{-1}(d_u/2)^{-2}/2 - \xi\mu c^2/2$$

accounting for the dynamic behavior of both $\mu$-particle and universe. Hence the energy balance per unit volume of universe consists of four terms: $\eta_u$, $\eta_{vac}$, $\eta_{zp}$ and $\delta\eta_{zp} = \delta\varepsilon/V_\mu$. The first two terms, equal by hypothesis, are also equal to the third by definition and have been already calculated; $\delta\varepsilon$ amounts to about $7.9\times10^{-53}$J, so that $\delta\eta_{zp} = 8.7\times10^{-10}$J/m$^3$. Hence $\delta\eta_{zp}$ is about 64% of $\delta\eta_{zp} + \eta_{vac}$ and about 35% of the total energy density $\delta\eta_{zp} + \eta_{vac} + \eta_u + \eta_{zp} = 2.4 \times 10^{-9}$J/m$^3$.

The former estimate is particularly interesting because neither $\eta_{vac}$ nor $\delta\eta_{zp}$ are directly related to the matter present in the universe; rather the picture so far outlined suggests that $\eta_{vac}$ is related to $\mu$ within $V_\mu$ randomly delocalized throughout the whole physical size of the universe, whereas the ordinary matter is in turn a local coalescence from the vacuum energy density precursor. This idea explains why $\mu' c^2/V_\mu = 1.1 \times 10^{-9}$Jm$^{-3}$ is twice $\eta_u$; actually this result must be intended as $\mu' c^2/V_\mu = \eta_{vac} + \eta_u$. As concerns the negative sign of $\mu$, see eqs. (4,3), note that actually the second eq (4,9) reads $\lambda_\mu = \pm\hbar/\mu c$ and that $\xi$ turns into $-\xi$ replacing $v$ with $-v$; it is easy to realize that this leaves unchanged $\lambda_\mu$ and the quantities that depend on $mu'$, e.g. $\omega_\mu$ and $V_\mu$, while the universe time $\tau$ of eq (4,9) changes its sign. Also $\sigma_\mu$ change its sign, so the tension $T$ must be replaced by $-T$.

The last remark concerns the physical meaning of $\delta\varepsilon$; it is neither vibrational or zero point energy of $\mu$, nor vacuum or matter energy. If so, what then is it? Is it the so called dark energy?

## 5 Discussion

The discussion of the results starts emphasizing the conceptual path followed in the previous sections to merge relativity and quantum physics via the basic eqs. (2,1). The prerequisites of the present model rest on three outstanding key words: quantization, non-locality, non-reality. Without sharing all three of these features together, the search of a unified theory would be physically unconvincing and intrinsically incomplete. The first result to be noted is that the present model of quantum relativity finds again formulae known since their early Einstein derivation, which indeed agree with the experimental results, although with a physical meaning actually different; instead of deterministic intervals, the relativistic formulae must be regarded as functions of the corresponding uncertainty ranges. On the one side, this coincidence ensures the consistency of the present theoretical model with the experience. On the other side, the sought unification unavoidably compels transferring the acknowledged weirdness of the quantum world to the relativistic phenomena: it requires regarding the intervals and distances likewise the ranges of eqs. (2,1), i.e. as a sort of evanescent entities, undefined and arbitrary, not specified or specifiable by any hypothesis, whose only feature and role rests on their conceptual existence and ability to replace the local dynamical variables, in no way defined and definable too. For instance the invariant interval of special relativity turns into a space-time uncertainty range whose size, whatever it might be, remains effectively unchanged in all inertial reference systems; in other words, this well known concept still holds despite its size is actually indeterminable.

Strictly speaking, it seems understandable that nothing else but an evanescent idea of uncertainty ranges could explain counterintuitive quantum features like the non-reality and non-locality; the former has been described in subsection 4.2 as a consequence of the measurement driven compliance of the eigenvalues with eqs. (2,1), the latter has been related in [9] to the elusiveness of concepts like local distances that hide the ultimate behavior of the matter. The EPR paradox or the dual corpuscle/wave behavior or the actual incompleteness of quantum mechanics testify in fact different appearances of the unique fundamental concept of uncertainty; the approach of sections 3 and 4 is so elementary and straightforward to suggest that the present way of reasoning focuses just on the limited degree of knowledge we can in fact afford, i.e. only on the physical outcome that waives any local information.

Despite this statement represents the most agnostic starting point possible, nevertheless it paradoxically connects quantum theory and relativity in the most profound way expectable: from their basic postulates to their most significant results. In this respect the section 4 shows an alternative conceptual path, less geometrical, towards some relevant outcomes of general relativity: Einstein's way to account for the gravity through the geometrical model of curved spacetime is replaced by simple considerations on the uncertainty ranges of four fundamental dynamical variables of eqs. (2,1). In this way the approach is intrinsically adherent to the quantum mechanics, which rests itself on the same equations. For this reason even the general relativity is compliant with the non-locality and non-reality of the quantum world, as it has been sketched in section 3.

This conclusion seems surprising, because usually the relativity aims to describe large objects on a cosmological scale; yet its features inferred in the present paper can be nothing else but a consequence of quantum properties consistent with well known formulae early conceived for other purposes. A more detailed and complete treatment is exposed in the paper [13], including also the gravitational waves and the perihelion precession of the Kepler problem.

The quantization of the gravity field is regarded as the major task in several relativistic models; although this idea is in principle reductive alone, because also the non-reality and non-locality deserve equal attention, examining the present results this way of thinking appears in fact acceptable. Indeed the number of states $n$ accounts not only for the quantization of the results, as it is obvious, but also for the non-locality and non-reality themselves; as highlighted in [9] the reality and locality of the classical world appear for $n \to \infty$ only, i.e. when $n$ tends to behave like a continuous variable so that the Bell inequality is fulfilled. So it is reasonable to think that the quantization has in effect a hierarchical role predominant on the other quantum properties. Yet this actually happens if $n$ is never exactly specified because of its arbitrariness, thus ensuring the invariancy of eqs. (2,1); its effectiveness in describing both quantum and relativistic worlds appears due indeed to its lack of specific definition and to its twofold meaning of number of states and quantum number. Just this ambivalence is the further feature that remarks the importance of $n$; on the one side it represents an essential outcome of the quantum mechanics, on the other side it assigns its quantum fingerprint to any macroscopic system necessarily characterized by a number of allowed states. Of course the incompleteness of information governing the quantum world compels an analogous limit to the relativity; yet, without accepting this restriction since the beginning into the sought unified model through eqs. (2,1), the elementary considerations of sections 3 and 4 would rise topmost difficulties in formulating correct outcomes. Moreover, typical ideas of quantum mechanics provide a possible explanation of experiments that involve relativistic concepts. An example in this respect has been proposed in the paper [9] as concerns the possibility of a superluminal velocity under investigation in a recent experiment carried out with neutrinos and still to be confirmed. A relativistic quantum fluctuation hypothesized in the quoted paper appears compatible with a superluminal velocity transient that, just because of its transitory character, can be justified without violating any standard result of the deterministic formulae of early relativity. Other problems are presently under investigation.

Regardless of the results still in progress, seems however significant "per se" the fact itself that the quantum character of the relativistic formulae widens in principle the descriptive applicability of the standard relativity.

Submitted on March 16, 2012 / Accepted on March 21, 2012

## References


1. Einstein A., Podolski B., Rosen N. Can quantum mechanics description of Physical Reality be considered Complete? *Physical Review*, 1935, v. 47, 777–780.
2. Polcinski J. String theory, Cambridge University Press, 1998, Cambridge.
3. Green M.B., Schwarz J.H. and Witten E. Superstring Theory, Cambridge University Press, (1987).
4. Carlip S., Quantum gravity: a progress report. *Reports on Progress in Physics*, 2001, v. 64, p. 885.
5. Becker K, Becker M., Schwarz J. String theory and M-Theory: a modern introduction. Cambridge University Press, Cambridge, 2007.
6. Wess J., Bagger J. Supersimmetry and Supergravity, Princeton University Press, Princeton, 1992.
7. Tosto S. An analysis of states in the phase space: the energy levels of quantum systems. *Il Nuovo Cimento B*, 1996, v. 111(2), 193–215.
8. Tosto S. An analysis of states in the phase space: the diatomic molecules. *Il Nuovo Cimento D*, 1996, v. 18(12), 1363–1394.
9. Tosto S. Spooky action at a distance or action at a spooky distance? *Progress in Physics*, 2012, v. 1, 11–26.
10. Tosto S. An analysis of states in the phase space: uncertainty, entropy and diffusion. *Progress in Physics*, 2011, v. 4, 68–78.
11. Landau L., Lifchits E. Theorie du Champ, MIR, Moscow, 1966.
12. Landau L., Lifchits E. Mechanique, MIR Moscow, 1969.
13. Tosto S. An analysis of states in the phase space: from quantum mechanics to general relativity, arXiv gr-qc/0807.1011.