1 Introduction

For fractals we refer to [1, 2] and for differential equations cf. also [3–7]. The theme of scale relativity as in [8–15] provides a profound development of differential calculus involving fractals (cf. also the work of Agop et al in the journal Chaos, Solitons, and Fractals) and for interaction with fractional calculus we mention [6,16–19]. There are also connections with the Riemann zeta function which we do not discuss here (see e.g. [20]). Now the recent paper [21] of Kobelev describes a Leibnitz type fractional derivative and one can relate fractional calculus with fractal structures as in [16, 18, 19, 25] for example. On the other hand scale relativity with Hausdorff dimension 2 is intimately related to the Schrödinger equation (SE) and quantum mechanics (QM) cf. [12]). We show now that if one can write a meaningful Schrödinger equation with Kobelev derivatives (α-derivatives) then there will be a corresponding fractional quantum potential (QP) (see e.g. [4, 6, 18, 19]) for a related fractional equation and recall that the classical wave function for the SE has the form $\psi = \exp(iS/\hbar)$.

Going now to [21] we recall the Riemann-Liouville (RL) type fractional operator (assumed to exist here)

\[ D^\alpha_\tau[f(z)] = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_\tau^z (z - \xi)^{\alpha-1} f(\xi)d\xi \\ c \in \mathbb{R}, \Re(\alpha) < 0 \\ \frac{d^m}{dz^m} D^{\alpha-m}[f(z)] \\ m - 1 \leq \Re(\alpha) < m \end{cases} \] (1.1)

(the latter for $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$). For $c = 0$ one writes (1A) $D^\alpha g(z) = D^\alpha[f(z)]$ as in the classical RL operator of order $\alpha$ (or $-\alpha$). Moreover when $c \to \infty$ (1.1) may be identified with the familiar Weyl fractional derivative (or integral) of order $\alpha$ (or $-\alpha$). An ordinary derivative corresponds to $\alpha = 1$ with (1B) $(d/dz)[f(z)] = D^1[f(z)]$. The binomial Leibnitz rule for derivatives is

\[ D^\alpha_\tau[g(z)f(z)] = g(z)D^\alpha_\tau[f(z)] + f(z)D^\alpha_\tau[g(z)] \] (1.2)

whose extension in terms of RL operators $D^\alpha_\tau$ has the form

\[ D^\alpha_\tau[f(z)g(z)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D^{\alpha-n}_\tau[f(z)]D^n_\tau[g(z)]; \] (1.3)

\[ \binom{\alpha}{n} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - k + 1)\Gamma(k + 1)}; \quad \alpha, k \in \mathbb{C}. \]

The infinite sum in (1.3) complicates things and the binomial Leibnitz rule of [21] will simplify things enormously. Thus consider first a momomial $z^\beta$ so that

\[ D^\alpha_\tau[z^\beta] = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} z^{\beta - \alpha}; \quad \Re(\alpha) < 0; \quad \Re(\beta) > -1. \] (1.4)

Thus the RL derivative of $z^\beta$ is the product

\[ D^\alpha_\tau[z^\beta] = C(\beta, \alpha)z^{\beta - \alpha}; \quad C(\beta, \alpha) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)}. \] (1.5)

Now one considers a new definition of a fractional derivative referred to as an $\alpha$ derivative in the form

\[ \frac{d_\alpha}{dz}[z^\beta] = d_\alpha[z^\beta] = C(\beta, \alpha)z^{\beta - \alpha}. \] (1.6)

This is required to satisfy the Leibnitz rule (1.2) by definition, given suitable conditions on $C(\beta, \alpha)$. Thus first (1C) $z^\beta = f(z)g(z)$ with $f(z) = z^{\beta - \epsilon}$ and $g(z) = z^\epsilon$ for arbitrary $\epsilon$ the application of (1.3) implies that

\[ \frac{d_\alpha}{dz}[z^\beta] = \frac{\epsilon}{\beta} \frac{d_\alpha}{dz}[z^{\beta - \epsilon}] + \frac{\beta - \epsilon}{\beta} \frac{d_\alpha}{dz}[z^\epsilon] \]

\[ = \frac{\epsilon}{\beta} C(\beta - \epsilon, \alpha)z^{\beta - \epsilon - \alpha} + \frac{\beta - \epsilon}{\beta} C(\epsilon, \alpha)z^{\beta - \alpha}. \] (1.7)

Comparison of (1.6) and (1.7) yields (1D) $C(\beta - \epsilon, \alpha) + C(\epsilon, \alpha) = C(\beta, \alpha)$. To guarantee (1.2) this must be satisfied for any $\beta, \epsilon, \alpha$. Thus (1D) is the basic functional equation and its solution is (1E) $C(\beta, \alpha) = A(\alpha)\beta$. Thus for the validity of the Leibnitz rule the $\alpha$-derivative must be of the form

\[ \frac{d_\alpha}{dz}[z^\beta] = \frac{d_\alpha}{dz}[z^\beta] = A(\alpha)\beta z^{\beta - \alpha}. \] (1.8)

One notes that $C(\beta, \alpha)$ in (1.5) is not of the form (1E) and the RL operator $D^\alpha_\tau$ does not in general possess a Leibnitz rule. One can assume now that $A(\alpha)$ is arbitrary and $A(\alpha) = 1$ is chosen. Consequently for any $\beta$

\[ \frac{d_\alpha}{dz}[z^\beta] = \beta z^{\beta - \alpha}; \quad \frac{d_\alpha}{dz}[z^\beta] = \alpha; \quad \frac{d_\alpha}{dz}[z^\beta] = 0. \] (1.9)

Now let $K$ denote an algebraically closed field of characteristic 0 with $K[x]$ the corresponding polynomial ring and...
K(x) the field of rational functions. Let F(z) have a Laurent series expansion about 0 of the form

\[ F(z) = \sum_{-\infty}^{\infty} c_k z^k; \]
\[ F_+(z) = \sum_{0}^{\infty} c_k z^k; \]
\[ F_-(z) = \sum_{-\infty}^{-1} c_k z^k; \quad c_k \in K \]

and generally there is a k₀ such that c_k = 0 for k ≤ k₀.

The standard ideas of differentiation hold for F(z) and formal power series form a ring K[[x]] with quotient field K((x)) (formal Laurent series). One considers now the union (1F) K ≪ x ≫ = \( \cup_{n=1}^{\infty} K((x^{1/n})) \). This becomes a field if we set

\[ x^{1/n} = x, \quad x^{m/n} = (x^{1/n})^m. \]

Then K ≪ x ≫ is called the field of fractional power series or the field of Puiseux series. If f ∈ K ≪ x ≫ has the form (1G) \( f = \sum_{k=1}^{\infty} c_k x^{k/n} \), where \( c_1 ≠ 0 \) and \( m, n ∈ \mathbb{N} \) \( (m/n) < (m'/n') \) for \( i < j \) then the order is (1H) \( O(f) = m/n \) where \( m = m_1 + m_2 \) and \( n = n_1 + n_2 \) and \( f(x) = F(x^{1/n}) \).

Now given n and x we look at functions

\[ f(z) = \sum_{-\infty}^{\infty} c_k (z - z_0)^{k/n}; \]
\[ f_+(z) = \sum_{0}^{\infty} c_k (z - z_0)^{k/n}; \]
\[ f_-(z) = \sum_{-\infty}^{-1} c_k (z - z_0)^{k/n}; \quad c_k = 0 \quad (k ≤ k₀) \]

(cf. [21] for more algebraic information - there are some misprints).

One considers next the α-derivative for a basis (1I) \( α = m/n; \quad 0 < m < n; \quad m, n ∈ \mathbb{N} \) \( (1, 2, 3, \cdots) \). The α-derivative of a Puiseux function of order \( O(f) = 1/n \) is again a Puiseux function of order \( (1 - m)/n \). For \( α = 1/n \) we have

\[ f_+ = \sum_{0}^{\infty} c_k z^{k/n}; \quad \beta = \beta(k) = k/n \]

leading to

\[ \frac{d_α}{dz} f_+(z) = \sum_{0}^{\infty} c_k \beta z^{k/n}; \quad \beta = \beta(k) = k/n \]
\[ \frac{d_α}{dz} f_-(z) = \sum_{-\infty}^{-1} c_k \beta z^{k/n}; \quad \beta = \beta(k) = k/n \]
\[ \frac{d_α}{dz} f_0(z) = \sum_{-\infty}^{\infty} c_k \beta z^{k/n}; \quad \beta = \beta(k) = k/n \]

(1.10)

(1.11)

(1.12)

(1.13)

(1.14)

(1.15)

(1.16)

(1.17)

(1.18)

(1.19)

(1.20)

Similar calculations hold for \( α = m/n \) (there are numerous typos and errors in indexing in [21] which we don’t mention further). The crucial property however is the Leibnitz rule

\[ \frac{d_α}{dz} f g = \frac{d_α}{dz} f + f \frac{d_α}{dz} g; \quad (d_α \sim \frac{d}{dz}) \]

which is proved via arguments with Puiseux functions. This leads to the important chain rule

\[ \frac{d_α}{dz} F(g(z)) = \sum \frac{∂F}{∂g_k} \frac{d_α}{dz} g_k(z). \]

Further calculation yields (again via use of Puiseux functions)

\[ \frac{d_α}{dz} \left[ \frac{d_α}{dz} f \right] = \frac{d_α}{dz} \left[ \frac{d_α}{dz} f \right] \]

(1.13)

(1.14)

(1.15)

(1.16)

(1.17)

(1.18)

(1.19)

(1.20)

The definition is motivated by the fact that \( E_α(z) \) satisfies the \( α \)-differential equation (1J) \( (d_α/dz)E_α(z) = E_α(z) \) with \( E_α(0) = 1 \). This is proved by term to term differentiation of (1.20). It is worth mentioning that \( E_α(z) \) does not possess the semigroup property (1K) \( E_α(z_1 + z_2) ≠ E_α(z_1)E_α(z_2) \).

2 Fractals and fractional calculus

For relations between fractals and fractional calculus we refer to [16, 18, 19, 24, 25, 27, 28]. In [16] for example one assumes time and space scale isotropically and writes \([x^α] = -1 \) for \( μ = 0, 1, \cdots, D - 1 \) and the standard measure is replaced by (2A) \( d^D x \rightarrow dp(x) \) with \( |p| = -Dα ≠ -D \) (note [ ] denotes the engineering dimension in momentum units). Here \( 0 < α < 1 \) is a parameter related to the operational definition of Hausdorff dimension which determines the scaling of a Euclidean volume (or mass distribution) of characteristic size R (i.e. \( V(R) \sim R^{Dα} \)). Taking \( α = \partial(R^{Dα}) \) one has (2B) \( V(R) \sim \int dp_{Euclid}(r) \sim R^{Dα} \) with \( \partial(R^{Dα}) \) showing that \( α = dH/D \). In general as cited in [16] the Hausdorff dimension of a random process (Brownian motion) described by a fractional differintegral is proportional to the order \( α \) of the differintegral. The same relation holds for deterministic fractals and in general the fractional differintegral of a curve
changes its Hausdorff dimension as $d_H \to d_H + \alpha$. Moreover integrals on “net fractals” can be approximated by the left sided RL fractional of a function $L(t)$ via

$$
\int_0^t \! dp(t)L(t) \approx \frac{1}{\Gamma(\alpha)} \int_0^t \! dt(\bar{t} - t)^{\alpha - 1} L(t); 
$$

(2.1)

$$
\rho(t) = \frac{\vec{p}^2 - (\bar{t} - t)^\alpha}{\Gamma(\alpha + 1)},
$$

where $\alpha$ is related to the Hausdorff dimension of the set (cf. [24]). Note that a change of variables $t \to \bar{t} - t$ transforms (2.1) to

$$
\frac{1}{\Gamma(\alpha)} \int_0^t \! dt^{\alpha - 1} \bar{L}(\bar{t} - t).
$$

(2.2)

The RL integral above can be mapped into a Weyl integral for $\bar{t} \to \infty$. Assuming $\lim_{\bar{t} \to \infty}$ the limit is formal if the Lagrangian $L$ is not autonomous and one assumes therefore that $\lim_{\bar{t} \to \infty} L(\bar{t} - t) = \hat{L}(\bar{t}, \bar{q})$ (leading to a Stieltjes field theory action). After constructing a “fractional phase space” this analogy confirms the interpretation of the order of the theory action. After constructing a “fractional phase space” one obtains a fractional integral for example. This means that the path integral based on the Feynman integrals on “net fractals” can be approximated by the left sided RL fractional of a function $L(t)$ via

$$
\int_0^t \! dp(t)L(t) \approx \frac{1}{\Gamma(\alpha)} \int_0^t \! dt(\bar{t} - t)^{\alpha - 1} L(t);
$$

(2.1)

$$
\rho(t) = \frac{\vec{p}^2 - (\bar{t} - t)^\alpha}{\Gamma(\alpha + 1)},
$$

with the standard inverse. Evidently (2.6) can be written in operator form as (2D) $i\hbar \partial t \psi = H_{\psi} \psi$ ; $H_{\psi} = -D_x(X^i) + V(x)$.

In [6] (0510099) a different approach is used involving the Caputo derivatives (where $\hat{D}^k(x) = k = 0$ for $k = constant$). Here for (2E) $f(x) = \int_0^\alpha \! a_n(x)^{\alpha n}$ one writes ($D \to \bar{D}$)

$$
\int_0^\alpha \! a_n(x)^{\alpha n}.
$$

(2.9)

Next to extend the definition to negative reals one writes

$$
\partial x \to \hat{x}(x) = \text{sgn}(x)|x|^\alpha; \hat{x}(x) = \text{sgn}(x)\bar{D}(x).
$$

(2.10)

There is a parity transformation II satisfying (2F) $\Pi \hat{f}(x) = -\hat{f}(x)$ and $\Pi \hat{x}(x) = \hat{x}(x)$. Then one defines (2G) $\hat{f}(\hat{x}(x)) = \int_0^\alpha \! a_n(x)^{\alpha n}(x)$ with a well defined derivative

$$
\hat{D}f(\hat{x}(x)) = \int_0^\alpha \! a_n(x)^{\alpha n}(x).
$$

(2.11)

This leads to a Hamiltonian $H^\alpha$ with

$$
H^\alpha = -\frac{1}{2} mc^2 \left( \frac{\hbar}{mc} \right) D^\alpha D + V(\hat{X}^1, \ldots, \hat{X}^N).
$$

(2.12)

with a time dependent SE

$$
\hat{H}^\alpha \Psi = \left[ \left. \frac{1}{2} mc^2 \left( \frac{\hbar}{mc} \right) D^\alpha D + V(\hat{X}^1, \ldots, \hat{X}^N) \right| \Psi \right.
$$

(2.13)

$$
= i\hbar \partial t \Psi.
$$

3 The SE with $\alpha$-derivative

Now we look at a 1-D SE with $\alpha$-derivatives $d_x \sim d_x/dx$ (without motivational physics). We write $d_x \phi = \beta x^{\alpha - 2}$ as in (1.9) and posit a candidate $\alpha$-SE in the form

$$
i\hbar \partial t \psi = D_x \hbar^2 \partial^\alpha_x \psi + V(x) \psi.
$$

(3.1)

In [11, 12] for example (cf. also [29]) one deals with a Schrödinger type equation

$$
\hat{D}^\alpha \hat{X} + i\hbar \partial t \hat{X} = \frac{\hbar^2}{2m} \hat{X} = 0
$$

(3.2)

where $\hat{D} \sim (\hbar/2m)$ in the quantum situation. Further $\hat{D}$ is allowed to have macro values with possible application in biology and cosmology (see Remark 3.1 below).

Consider a possible solution corresponding to $\psi = R \exp(iS/\hbar)$ in the form (3A) $\psi = R \exp(iS/\hbar)$ with $E_\alpha$ as in (1.20). Then one has for $S = S(x, t)$ (3B) $\psi = R(x, t) + \hat{R} \partial_t E_\alpha$ and via (1.15)-(1.16)

$$
d_x \left[ \int E_\alpha \left( \frac{iS}{\hbar} \right) \right] = (d_x R) E_\alpha + \int E_\alpha \left( \frac{1}{\hbar} \right) (d_x S);
$$

(3.3)

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We note that the techniques of scale relativity (cf. [11, 12]) have macro values.

For the classical case with \(d_R \approx \alpha = 1\) one has \(D_a = 1/2m\) and one imagines more generally that \(D_a \hbar^2\) may have macro values.

**Remark 3.1**

We note that the techniques of scale relativity (cf. [11, 12]) lead to quantum mechanics (QM). In the non-relativistic case the fractal Hausdorff dimension \(d_H = 2\) arises and one can generate the standard quantum potential (QP) directly (cf. also [29]). The QP turns out to be a critical factor in understanding QM (cf. [30–32, 35–37]) while various macro versions of QM have been suggested in biology, cosmology, etc. (cf. [8, 11, 12, 38, 39]). The sign of the QP serves to distinguish diffusion from an equation with a structure forming energy term (namely QM for \(D_a = 1/2m\) and fractal paths of Hausdorff dimension 2). The multi-fractal universe of [16,23] can involve fractional calculus with various degrees \(\alpha\) (i.e. fractals of differing Hausdorff dimension). We have shown that, given a physical input for (3.1) with the \(\alpha\)-derivative of \(K_0\) (21), the accompanying \(\alpha\)-QP could be related to structure formation in the related theory.

**References**


Robert Carroll. On a Fractional Quantum Potential