On the Decomposition of the Spacetime Metric Tensor and of Tensor Fields in Strained Spacetime

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We propose a natural decomposition of the spacetime metric tensor of General Relativity into a background and a dynamical part based on an analysis from first principles of the effect of a test mass on the background metric. We find that the presence of mass results in strains in the spacetime continuum. Those strains correspond to the dynamical part of the spacetime metric tensor. We then apply the stress-strain relation of Continuum Mechanics to the spacetime continuum to show that rest-mass energy density arises from the volume dilatation of the spacetime continuum. Finally we propose a natural decomposition of tensor fields in strained spacetime, in terms of dilatations and distortions. We show that dilatations correspond to rest-mass energy density, while distortions correspond to massless shear transverse waves. We note that this decomposition in a massive dilatation and a massless transverse wave distortion, where both are present in spacetime continuum deformations, is somewhat reminiscent of wave-particle duality. We note that these results are considered to be local effects in the particular reference frame of the observer. In addition, the applicability of the proposed metric to the Einstein field equations remains open.

1 Introduction

We first demonstrate from first principles that spacetime is strained by the presence of mass. Strained spacetime has been explored recently by Tartaglia *et al.* in the cosmological context, as an extension of the spacetime Lagrangian to obtain a generalized Einstein equation [1, 2]. Instead, in this analysis, we consider strained spacetime within the framework of Continuum Mechanics and General Relativity. This allows for the application of continuum mechanical results to the spacetime continuum. In particular, this provides a natural decomposition of the spacetime metric tensor and of spacetime tensor fields, both of which are still unresolved and are the subject of continuing investigations (see for example [3–7]).

2 Decomposition of the Spacetime Metric Tensor

There is no straightforward definition of local energy density of the gravitational field in General Relativity [8, see p. 84, p. 286] [6, 9, 10]. This arises because the spacetime metric tensor includes both the background spacetime metric and the local dynamical effects of the gravitational field. No natural way of decomposing the spacetime metric tensor into its background and dynamical parts is known.

In this section, we propose a natural decomposition of the spacetime metric tensor into a background and a dynamical part. This is derived from first principles by introducing a test mass in the spacetime continuum described by the background metric, and calculating the effect of this test mass on the metric.

Consider the diagram of Figure 1. Points A and B of the spacetime continuum, with coordinates x^μ and $x^\mu + dx^\mu$ respectively, are separated by the infinitesimal line element

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor describing the background state of the spacetime continuum.

We now introduce a test mass in the spacetime continuum. This results in the displacement of point A to $\widetilde{A}$, where the displacement is written as u^μ. Similarly, the displacement of point B to $\widetilde{B}$ is written as $u^\mu + du^\mu$. The infinitesimal line element between points $\widetilde{A}$ and $\widetilde{B}$ is given by $\widetilde{ds}^2$.

By reference to Figure 1, the infinitesimal line element $\widetilde{ds}^2$ can be expressed in terms of the background metric tensor as

$$\widetilde{ds}^2 = g_{\mu\nu}(dx^\mu + du^\mu)(dx^\nu + du^\nu). \tag{2}$$

Multiplying out the terms in parentheses, we get

$$\widetilde{ds}^2 = g_{\mu\nu}(dx^\mu dx^\nu + dx^\mu du^\nu + du^\mu dx^\nu + du^\mu du^\nu). \tag{3}$$

Expressing the differentials du as a function of x, this equation becomes

$$\begin{aligned}\widetilde{ds}^2 = g_{\mu\nu}(dx^\mu dx^\nu + dx^\mu\, u^\nu{}_{;\alpha}\, dx^\alpha + u^\mu{}_{;\alpha}\, dx^\alpha dx^\nu + \\ + u^\mu{}_{;\alpha}\, dx^\alpha\, u^\nu{}_{;\beta}\, dx^\beta)\end{aligned} \tag{4}$$

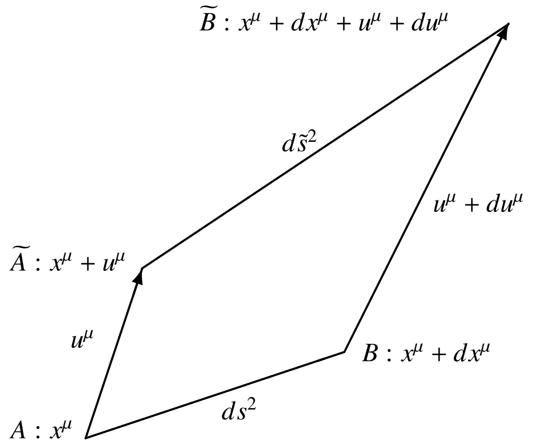
where the semicolon (;) denotes covariant differentiation. Rearranging the dummy indices, this expression can be written as

$$\widetilde{ds}^2 = (g_{\mu\nu} + g_{\mu\alpha}\, u^\alpha{}_{;\nu} + g_{\alpha\nu}\, u^\alpha{}_{;\mu} + g_{\alpha\beta}\, u^\alpha{}_{;\mu} u^\beta{}_{;\nu})\, dx^\mu dx^\nu \tag{5}$$

and lowering indices, the equation becomes

$$\widetilde{ds}^2 = (g_{\mu\nu} + u_{\mu;\nu} + u_{\nu;\mu} + u^\alpha{}_{;\mu} u_{\alpha;\nu})\, dx^\mu dx^\nu. \tag{6}$$

Fig. 1: Effect of a test mass on the background metric tensor



The expression $u_{\mu;\nu} + u_{\nu;\mu} + u^\alpha{}_{;\mu} u_{\alpha;\nu}$ is equivalent to the definition of the strain tensor $\varepsilon^{\mu\nu}$ of Continuum Mechanics. The strain $\varepsilon^{\mu\nu}$ is expressed in terms of the displacements u^μ of a continuum through the kinematic relation [11, see p. 149] [12, see pp. 23–28]:

$$\varepsilon^{\mu\nu} = \frac{1}{2}(u^{\mu;\nu} + u^{\nu;\mu} + u^{\alpha;\mu} u_\alpha{}^{;\nu}). \tag{7}$$

Substituting for $\varepsilon^{\mu\nu}$ from Eq.(7) into Eq.(6), we get

$$\widetilde{ds}^2 = (g_{\mu\nu} + 2\,\varepsilon_{\mu\nu})\, dx^\mu dx^\nu. \tag{8}$$

Setting [12, see p. 24]

$$\widetilde{g}_{\mu\nu} = g_{\mu\nu} + 2\,\varepsilon_{\mu\nu} \tag{9}$$

then Eq.(8) becomes

$$\widetilde{ds}^2 = \widetilde{g}_{\mu\nu}\, dx^\mu dx^\nu \tag{10}$$

where $\widetilde{g}_{\mu\nu}$ is the metric tensor describing the spacetime continuum with the test mass.

Given that $g_{\mu\nu}$ is the background metric tensor describing the background state of the continuum, and $\widetilde{g}_{\mu\nu}$ is the spacetime metric tensor describing the final state of the continuum with the test mass, then $2\,\varepsilon_{\mu\nu}$ must represent the dynamical part of the spacetime metric tensor due to the test mass:

$$g^{dyn}_{\mu\nu} = 2\,\varepsilon_{\mu\nu}. \tag{11}$$

We are thus led to the conclusion that the presence of mass results in strains in the spacetime continuum. Those strains correspond to the dynamical part of the spacetime metric tensor. Hence the applied stresses from mass (i.e. the energy-momentum stress tensor) result in strains in the spacetime continuum, that is strained spacetime.

3 Rest-Mass Energy Relation

The introduction of strains in the spacetime continuum as a result of the energy-momentum stress tensor allows us to use by analogy results from Continuum Mechanics, in particular the stress-strain relation, to provide a better understanding of strained spacetime.

The stress-strain relation for an isotropic and homogeneous spacetime continuum can be written as [12, see pp. 50–53]:

$$2\mu_0 \varepsilon^{\mu\nu} + \lambda_0 g^{\mu\nu} \varepsilon = T^{\mu\nu} \tag{12}$$

where $T^{\mu\nu}$ is the energy-momentum stress tensor, $\varepsilon^{\mu\nu}$ is the resulting strain tensor, and

$$\varepsilon = \varepsilon^\alpha{}_\alpha \tag{13}$$

is the trace of the strain tensor obtained by contraction. ε is the volume dilatation defined as the change in volume per original volume [11, see p. 149–152] and is an invariant of the strain tensor. λ_0 and μ_0 are the Lamé elastic constants of the spacetime continuum: μ_0 is the shear modulus and λ_0 is expressed in terms of κ_0, the bulk modulus:

$$\lambda_0 = \kappa_0 - \mu_0/2 \tag{14}$$

in a four-dimensional continuum. The contraction of Eq.(12) yields the relation

$$2(\mu_0 + 2\lambda_0)\varepsilon = T^\alpha{}_\alpha \equiv T. \tag{15}$$

The time-time component T^{00} of the energy-momentum stress tensor represents the total energy density given by [13, see pp. 37–41]

$$T^{00}(x^k) = \int d^3\mathbf{p} E_p f(x^k, \mathbf{p}) \tag{16}$$

where $E_p = (\rho^2 c^4 + p^2 c^2)^{1/2}$, ρ is the rest-mass energy density, c is the speed of light, $\mathbf{p}$ is the momentum 3-vector and $f(x^k, \mathbf{p})$ is the distribution function representing the number of particles in a small phase space volume $d^3\mathbf{x} d^3\mathbf{p}$. The space-space components T^{ij} of the energy-momentum stress tensor represent the stresses within the medium given by

$$T^{ij}(x^k) = c^2 \int d^3\mathbf{p} \frac{p^i p^j}{E_p} f(x^k, \mathbf{p}). \tag{17}$$

They are the components of the net force acting across a unit area of a surface, across the x^i planes in the case where $i = j$.

In the simple case of a particle, they are given by [14, see p. 117]

$$T^{ii} = \rho\, v^i v^i \tag{18}$$

where v^i are the spatial components of velocity. If the particles are subject to forces, these stresses must be included in the energy-momentum stress tensor.

Explicitly separating the time-time and the space-space components, the trace of the energy-momentum stress tensor is written as

$$T^\alpha{}_\alpha = T^0{}_0 + T^i{}_i. \tag{19}$$

Substituting from Eq.(16) and Eq.(17), using the metric $\eta^{\mu\nu}$ of signature (+ - - -), we obtain:

$$T^\alpha{}_\alpha(x^k) = \int d^3\mathbf{p}\left(E_p - \frac{p^2c^2}{E_p}\right) f(x^k,\mathbf{p}) \tag{20}$$

which simplifies to

$$T^\alpha{}_\alpha(x^k) = \rho^2 c^4 \int d^3\mathbf{p}\frac{f(x^k,\mathbf{p})}{E_p}. \tag{21}$$

Using the relation [13, see p. 37]

$$\frac{1}{\overline{E}_{har}(x^k)} = \int d^3\mathbf{p}\frac{f(x^k,\mathbf{p})}{E_p} \tag{22}$$

in equation Eq.(21), we obtain the relation

$$T^\alpha{}_\alpha(x^k) = \frac{\rho^2c^4}{\overline{E}_{har}(x^k)} \tag{23}$$

where $\overline{E}_{har}(x^k)$ is the Lorentz invariant harmonic mean of the energy of the particles at x^k.

In the harmonic mean of the energy of the particles $\overline{E}_{har}$, the momentum contribution $\mathbf{p}$ will tend to average out and be dominated by the mass term ρc^2, so that we can write

$$\overline{E}_{har}(x^k) \simeq \rho c^2. \tag{24}$$

Substituting for $\overline{E}_{har}$ in Eq.(23), we obtain the relation

$$T^\alpha{}_\alpha(x^k) \simeq \rho c^2. \tag{25}$$

The total rest-mass energy density of the system is obtained by integrating over all space:

$$T^\alpha{}_\alpha = \int d^3\mathbf{x}\; T^\alpha{}_\alpha(x^k). \tag{26}$$

The expression for the trace derived from Eq.(19) depends on the composition of the sources of the gravitational field. Considering the energy-momentum stress tensor of the electromagnetic field, we can show that $T^\alpha{}_\alpha = 0$ as expected for massless photons, while

$$T^{00} = \frac{\epsilon_0}{2}\left(E^2 + c^2B^2\right)$$

is the total energy density, where ϵ_0 is the electromagnetic permittivity of free space, and E and B have their usual significance.

Hence $T^\alpha{}_\alpha$ corresponds to the invariant rest-mass energy density and we write

$$T^\alpha{}_\alpha = T = \rho c^2 \tag{27}$$

where ρ is the rest-mass energy density. Using Eq.(27) into Eq.(15), the relation between the invariant volume dilatation ε and the invariant rest-mass energy density becomes

$$2(\mu_0 + 2\lambda_0)\varepsilon = \rho c^2 \tag{28}$$

or, in terms of the bulk modulus κ_0,

$$4\kappa_0\varepsilon = \rho c^2. \tag{29}$$

This equation demonstrates that rest-mass energy density arises from the volume dilatation of the spacetime continuum. The rest-mass energy is equivalent to the energy required to dilate the volume of the spacetime continuum, and is a measure of the energy stored in the spacetime continuum as volume dilatation. κ_0 represents the resistance of the spacetime continuum to dilatation. The volume dilatation is an invariant, as is the rest-mass energy density.

4 Decomposition of Tensor Fields in Strained Spacetime

As opposed to vector fields which can be decomposed into longitudinal (irrotational) and transverse (solenoidal) components using the Helmholtz representation theorem [11, see pp. 260–261], the decomposition of spacetime tensor fields can be done in many ways (see for example [3–5, 7]).

The application of Continuum Mechanics to a strained spacetime continuum offers a natural decomposition of tensor fields, in terms of dilatations and distortions [12, see pp. 58–60]. A *dilatation* corresponds to a change of volume of the spacetime continuum without a change of shape (as seen in Section 3) while a *distortion* corresponds to a change of shape of the spacetime continuum without a change in volume. Dilatations correspond to longitudinal displacements and distortions correspond to transverse displacements [11, see p. 260].

The strain tensor $\varepsilon^{\mu\nu}$ can thus be decomposed into a strain deviation tensor $e^{\mu\nu}$ (the *distortion*) and a scalar e (the *dilatation*) according to [12, see pp. 58–60]:

$$\varepsilon^{\mu\nu} = e^{\mu\nu} + e g^{\mu\nu} \tag{30}$$

where

$$e^\mu{}_\nu = \varepsilon^\mu{}_\nu - e\delta^\mu{}_\nu \tag{31}$$

$$e = \frac{1}{4}\varepsilon^\alpha{}_\alpha = \frac{1}{4}\varepsilon. \tag{32}$$

Similarly, the energy-momentum stress tensor $T^{\mu\nu}$ is decomposed into a stress deviation tensor $t^{\mu\nu}$ and a scalar t according to

$$T^{\mu\nu} = t^{\mu\nu} + t g^{\mu\nu} \tag{33}$$

where similarly

$$t^\mu{}_\nu = T^\mu{}_\nu - t\delta^\mu{}_\nu \tag{34}$$

$$t = \frac{1}{4}T^\alpha{}_\alpha. \tag{35}$$

Using Eq.(30) to Eq.(35) into the strain-stress relation of Eq.(12) and making use of Eq.(15) and Eq.(14), we obtain separated dilatation and distortion relations respectively:

$$\begin{aligned} &\text{dilatation}: t = 2(\mu_0 + 2\lambda_0)e = 4\kappa_0 e = \kappa_0 \varepsilon \\ &\text{distortion}: t^{\mu\nu} = 2\mu_0 e^{\mu\nu}. \end{aligned} \tag{36}$$

The distortion-dilatation decomposition is evident in the dependence of the dilatation relation on the bulk modulus κ_0 and of the distortion relation on the shear modulus μ_0. As shown in Section 3, the dilatation relation of Eq.(36) corresponds to rest-mass energy, while the distortion relation is traceless and thus massless, and corresponds to shear transverse waves.

This decomposition in a massive dilatation and a massless transverse wave distortion, where both are present in spacetime continuum deformations, is somewhat reminiscent of wave-particle duality. This could explain why dilatation-measuring apparatus measure the massive 'particle' properties of the deformation, while distortion-measuring apparatus measure the massless transverse 'wave' properties of the deformation.

5 Conclusion

In this paper, we have proposed a natural decomposition of the spacetime metric tensor into a background and a dynamical part based on an analysis from first principles, of the impact of introducing a test mass in the spacetime continuum. We have found that the presence of mass results in strains in the spacetime continuum. Those strains correspond to the dynamical part of the spacetime metric tensor.

We have applied the stress-strain relation of Continuum Mechanics to the spacetime continuum to show that rest-mass energy density arises from the volume dilatation of the spacetime continuum.

Finally we have proposed a natural decomposition of tensor fields in strained spacetime, in terms of dilatations and distortions. We have shown that dilatations correspond to rest-mass energy density, while distortions correspond to massless shear transverse waves. We have noted that this decomposition in a dilatation with rest-mass energy density and a massless transverse wave distortion, where both are simultaneously present in spacetime continuum deformations, is somewhat reminiscent of wave-particle duality.

It should be noted that these results are considered to be local effects in the particular reference frame of the observer. In addition, the applicability of the proposed metric to the Einstein field equations remains open.

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