

Should the Wave-Function be a Part of the Quantum Ontological State?

Aurélien Drezet

Institut Neel, 25 rue des Martyrs 38042, Grenoble, France. E-mail: aurelien.drezet@grenoble.cnrs.fr

We analyze the recent no go theorem by Pusey, Barrett and Rudolph (PBR) concerning ontic and epistemic hidden variables. We define two fundamental requirements for the validity of the result. We finally compare the models satisfying the theorem with the historical hidden variable approach proposed by de Broglie and Bohm.

1 Introduction

Recently, a new no go theorem by M. Pusey, J. Barret and T. Rudolph (PBR in the following) was published [1]. The result concerns ontic versus epistemic interpretations of quantum mechanics. Epistemic means here knowledge by opposition to “ontic” or ontological and is connected with the statistical interpretation defended by Einstein. This of course stirred much debates and discussions to define the condition of validity of this fundamental theorem. Here, we discuss two fundamental requirements necessary for the demonstration of the result and also discuss the impact of the result on possible hidden variable models. In particular, we will stress the difference between the models satisfying the PBR theorem and those who apparently contradict its generality.

2 The axioms of the PBR theorem

In order to identify the main assumptions and conclusions of the PBR theorem we first briefly restate the original reasoning of ref. 1 in a slightly different language. In the simplest version PBR considered two non orthogonal pure quantum states $|\Psi_1\rangle = |0\rangle$ and $|\Psi_2\rangle = [|0\rangle + |1\rangle]/\sqrt{2}$ belonging to a 2-dimensional Hilbert space \mathbb{E} with basis vectors $\{|0\rangle, |1\rangle\}$. Using a specific (nonlocal) measurement M with basis $|\xi_i\rangle$ ($i \in [1, 2, 3, 4]$) in $\mathbb{E} \otimes \mathbb{E}$ (see their equation 1 in [1]) they deduced that $\langle \xi_1 | \Psi_1 \otimes \Psi_1 \rangle = \langle \xi_2 | \Psi_1 \otimes \Psi_2 \rangle = \langle \xi_3 | \Psi_2 \otimes \Psi_1 \rangle = \langle \xi_4 | \Psi_2 \otimes \Psi_2 \rangle = 0$. In a second step they introduced hypothetical “Bell’s like” hidden variables λ and wrote implicitly the probability of occurrence $P_M(\xi_i; j, k) = |\langle \xi_i | \Psi_j \otimes \Psi_k \rangle|^2$ in the form:

$$P_M(\xi_i; j, k) = \int P_M(\xi_i | \lambda, \lambda') \varrho_j(\lambda) \varrho_k(\lambda') d\lambda d\lambda' \quad (1)$$

where $i \in [1, 2, 3, 4]$ and $j, k \in [1, 2]$. One of the fundamental axiom used by PBR (axiom 1) is an independence criterion at the preparation which reads $\varrho_{j,k}(\lambda, \lambda') = \varrho_j(\lambda) \varrho_k(\lambda')$. In these equations we introduced the conditional “transition” probabilities $P_M(\xi_i | \lambda, \lambda')$ for the outcomes ξ_i supposing the hidden state λ, λ' associated with the two independent Q-bits are given. The fundamental point here is that $P_M(\xi_i | \lambda, \lambda')$ is independent of Ψ_1, Ψ_2 . This a very natural looking-like axiom (axiom 2) which was implicit in ref. 1 and was not further discussed by the authors. We will see later what are the consequence of its abandonment.

For now, from the definitions and axioms we obtain:

$$\left. \begin{aligned} \int P_M(\xi_1 | \lambda, \lambda') \varrho_1(\lambda) \varrho_1(\lambda') d\lambda d\lambda' &= 0 \\ \int P_M(\xi_2 | \lambda, \lambda') \varrho_1(\lambda) \varrho_2(\lambda') d\lambda d\lambda' &= 0 \\ \int P_M(\xi_3 | \lambda, \lambda') \varrho_2(\lambda) \varrho_1(\lambda') d\lambda d\lambda' &= 0 \\ \int P_M(\xi_4 | \lambda, \lambda') \varrho_2(\lambda) \varrho_2(\lambda') d\lambda d\lambda' &= 0 \end{aligned} \right\} \quad (2)$$

The first line implies $P_M(\xi_1 | \lambda, \lambda') = 0$ if $\varrho_1(\lambda) \varrho_1(\lambda') \neq 0$. This condition is always satisfied if λ and λ' are in the support of ϱ_1 in the λ -space and λ' -space. Similarly, the fourth line implies $P_M(\xi_4 | \lambda, \lambda') = 0$ if $\varrho_2(\lambda) \varrho_2(\lambda') \neq 0$ which is again always satisfied if λ and λ' are in the support of ϱ_2 in the λ -space and λ' -space. Finally, the second and third lines imply $P_M(\xi_2 | \lambda, \lambda') = 0$ if $\varrho_1(\lambda) \varrho_2(\lambda') \neq 0$ and $P_M(\xi_3 | \lambda, \lambda') = 0$ if $\varrho_1(\lambda) \varrho_2(\lambda') \neq 0$.

Taken separately these four conditions are not problematic. But, in order to be true simultaneously and then to have

$$P_M(\xi_i | \lambda, \lambda') = 0 \quad (3)$$

for a same pair of λ, λ' (with $[i = 1, 2, 3, 4]$) the conditions require that the supports of ϱ_1 and ϱ_2 intersect. If this is the case Eq. 3 will be true for any pair λ, λ' in the intersection.

However, this is impossible since from probability conservation we must have $\sum_{i=1}^4 P_M(\xi_i | \lambda, \lambda') = 1$ for every pair λ, λ' . Therefore, we must necessarily have

$$\varrho_2(\lambda) \cdot \varrho_1(\lambda) = 0 \quad \forall \lambda \quad (4)$$

i.e. that ϱ_1 and ϱ_2 have nonintersecting supports in the λ -space. Indeed, it is then obvious to see that Eq. 2 is satisfied if Eq. 4 is true. This constitutes the PBR theorem for the particular case of independent prepared states Ψ_1, Ψ_2 defined before. PBR generalized their results for more arbitrary states using similar and astute procedures described in ref. 1.

If this theorem is true it would apparently make hidden variables completely redundant since it would be always possible to define a bijection or relation of equivalence between the λ space and the Hilbert space: (loosely speaking we could in principle make the correspondence $\lambda \leftrightarrow \psi$). Therefore it would be as if λ is nothing but a new name for Ψ itself. This would justify the label “ontic” given to this kind of interpretation in opposition to “epistemic” interpretations ruled out by the PBR result.

However, the PBR conclusion stated like that is too strong as it can be shown by carefully examining the assumptions necessary for the derivation of the theorem. Indeed, using the independence criterion and the well known Bayes-Laplace formula for conditional probability we deduce that the most general Bell's hidden variable probability space should obey the following rule

$$P_M(\xi_i; j, k) = \int P_M(\xi_i|\Psi_j, \Psi_k, \lambda, \lambda') \varrho_j(\lambda) \varrho_k(\lambda') d\lambda d\lambda' \quad (5)$$

in which, in contrast to equation 1, the transition probabilities $P_M(\xi_i|\Psi_j, \Psi_k, \lambda, \lambda')$ now depend explicitly on the considered quantum states Ψ_j, Ψ_k . We point out that unlike λ , Ψ is in this more general approach not a stochastic variable. This difference is particularly clear in the ontological interpretation of ref. 3 where Ψ plays the role of a dynamic guiding wave for the stochastic motion of the particle. Clearly, relaxing this PBR premise has a direct effect since we lose the ingredient necessary for the demonstration of Eq. 4. (more precisely we are no longer allowed to compare the product states $|\Psi_j \otimes \Psi_k\rangle$ as it was done in ref. 1). Indeed, in order for Eq. 2 to be simultaneously true for the four states ξ_i (where $P_M(\xi_i|\Psi_j, \Psi_k, \lambda, \lambda')$ now replace $P_M(\xi_i|\lambda, \lambda')$) we must have

$$\left. \begin{aligned} P_M(\xi_1|\Psi_1, \Psi_1, \lambda, \lambda') = 0, P_M(\xi_2|\Psi_1, \Psi_2, \lambda, \lambda') = 0 \\ P_M(\xi_3|\Psi_2, \Psi_1, \lambda, \lambda') = 0, P_M(\xi_4|\Psi_2, \Psi_2, \lambda, \lambda') = 0 \end{aligned} \right\}. \quad (6)$$

Obviously, due to the explicit Ψ dependencies, Eq. 6 doesn't anymore enter in conflict with the conservation probability rule and therefore doesn't imply Eq. 4. In other words the reasoning leading to PBR theorem doesn't run if we abandon the axiom stating that

$$P_M(\xi_i|\Psi_j, \Psi_k, \lambda, \lambda') := P_M(\xi_i|\lambda, \lambda') \quad (7)$$

i.e. that the dynamic should be independent of Ψ_1, Ψ_2 . This analysis clearly shows that Eq. 7 is a fundamental prerequisite (as important as the independence criterion at the preparation) for the validity of the PBR theorem [4]. In our knowledge this point was not yet discussed [5].

3 Discussion

Therefore, the PBR deduction presented in ref. 1 is actually limited to a very specific class of Ψ -epistemic interpretations. It fits well with the XIXth like hidden variable models using Liouville and Boltzmann approaches (i.e. models where the transition probabilities are independent of Ψ) but it is not in agreement with neo-classical interpretations, e.g. the one proposed by de Broglie and Bohm [3], in which the transition probabilities $P_M(\xi|\lambda, \Psi)$ and the trajectories depend explicitly and contextually on the quantum states Ψ (the de Broglie-Bohm theory being deterministic these probabilities can only reach values 0 or 1 for discrete observables ξ). As an illustration, in the de Broglie Bohm model for a single particle the

spatial position \mathbf{x} plays the role of λ . This model doesn't require the condition $\varrho_1(\lambda) \cdot \varrho_2(\lambda) = |\langle \mathbf{x}|\Psi_1\rangle|^2 \cdot |\langle \mathbf{x}|\Psi_1\rangle|^2 = 0$ for all λ in clear contradiction with Eq. 4. We point out that our reasoning doesn't contradict the PBR theorem *per se* since the central axiom associated with Eq. 7 is not true anymore for the model considered. In other words, if we recognize the importance of the second axiom discussed before (i.e. Eq. 7) the PBR theorem becomes a general result which can be stated like that:

i) If Eq. 7 applies then the deduction presented in ref. 1 shows that Eq. 4 results and therefore $\lambda \leftrightarrow \Psi$ which means that epistemic interpretation of Ψ are equivalent to ontic interpretations. This means that a XIXth like hidden variable models is not really possible even if we accept Eq. 7 since we don't have any freedom on the hidden variable density $\rho(\lambda)$.

ii) However, if Eq. 7 doesn't apply then the ontic state of the wavefunction is already assumed - because it is a variable used in the definition of $P_M(\xi|\lambda, \Psi)$. This shows that ontic interpretation of Ψ is necessary. This is exemplified in the de Broglie-Bohm example: in this model, the "quantum potential" is assumed to be a real physical field which depends on the magnitude of the wavefunction, while the motion of the Bohm particle depends on the wavefunction's phase. This means that the wavefunction has ontological status in such a theory. This is consistent with the spirit of PBR's paper, but the authors didn't discussed that fundamental point.

We also point out that in the de Broglie-Bohm ontological approach the independence criterion at the preparation is respected in the regime considered by PBR. As a consequence, it is not needed to invoke retrocausality to save epistemic approaches.

It is important to stress how Eq. 4, which is a consequence of Eq. 7, contradicts the spirit of most hidden variable approaches. Consider indeed, a wave packet which is split into two well spatially localized waves Ψ_1 and Ψ_2 defined in two isolated regions 1 and 2. Now, the experimentalist having access to local measurements ξ_1 in region 1 can define probabilities $|\langle \xi_1|\Psi_1\rangle|^2$. In agreement with de Broglie and Bohm most proponents of hidden variables would now say that the hidden variable λ of the system actually present in box 1 should not depends of the overall phase existing between Ψ_1 and Ψ_2 . In particular the density of hidden variables $\varrho_\Psi(\lambda)$ in region 1 should be the same for $\Psi = \Psi_1 + \Psi_2$ and $\Psi' = \Psi_1 - \Psi_2$ since $|\langle \xi_1|\Psi\rangle|^2 = |\langle \xi_1|\Psi'\rangle|^2$ for every local measurements ξ_1 in region 1. This is a weak form of separability which is accepted even within the so exotic de Broglie Bohm's approach but which is rejected for those models accepting Eq. 4.

This point can be stated differently. Considering the state $\Psi = \Psi_1 + \Psi_2$ previously discussed we can imagine a two-slits like interference experiment in which the probability for detecting outcomes x_0 , i.e., $|\langle x_0|\Psi\rangle|^2$ vanish for some values x_0 while $|\langle x_0|\Psi_1\rangle|^2$ do not. For those models satisfying Eq. 7 and forgetting one instant PBR theorem we deduce that in the hypothetical common support of $\varrho_{\Psi_1}(\lambda)$ and $\varrho_\Psi(\lambda)$ we must have

$P_M(\xi_0|\lambda) = 0$ since this transition probability should vanish in the support of Ψ . This allows us to present a “*poor-man*” version of the PBR’s theorem: The support of $\varrho_{\Psi_1}(\lambda)$ can not be completely included in the support of $\varrho_{\Psi}(\lambda)$ since otherwise $P_M(\xi_0|\lambda) = 0$ would implies $|\langle x_0|\Psi_1\rangle|^2 = 0$ in contradiction with the definition. PBR’s theorem is stronger than that since it shows that in the limit of validity of Eq. 7 the support of $\varrho_{\Psi_1}(\lambda)$ and $\varrho_{\Psi}(\lambda)$ are necessarily disjoint. Consequently, for those particular models the hidden variables involved in the observation of the observable ξ_0 are not the same for the two states Ψ and Ψ_1 . This is fundamentally different from de Broglie-Bohm approach where λ (e.g. $\mathbf{x}(t_0)$) can be the same for both states.

This can lead to an interesting form of quantum correlation even with one single particle. Indeed, following the well known scheme of the Wheeler Gedanken experiment one is free at the last moment to either observe the interference pattern (i.e. $|\langle x_0|\Psi\rangle|^2 = 0$) or to block the path 2 and destroy the interference (i.e. $|\langle x_0|\Psi_1\rangle|^2 = 1/2$). In the model used by Bohm where Ψ acts as a guiding or pilot wave this is not surprising: blocking the path 2 induces a subsequent change in the propagation of the pilot wave which in turn affects the particle trajectories. Therefore, the trajectories will not be the same in these two experiments and there is no paradox. However, in the models considered by PBR there is no guiding wave since Ψ serves only to label the non overlapping density functions of hidden variable $\varrho_{\Psi_1}(\lambda)$ and $\varrho_{\Psi}(\lambda)$. Since the beam block can be positioned after the particles leaved the source the hidden variable are already predefined (i.e. they are in the support of $\varrho_{\Psi}(\lambda)$). Therefore, the trajectories are also predefined in those models and we apparently reach a contradiction since we should have $P_M(\xi_0|\lambda) = 0$ while we experimentally record particles with properties ξ_0 . The only way to solve the paradox is to suppose that some mysterious quantum influence is sent from the beam blocker to the particle in order to modify the path during the propagation and correlate it with presence or absence of the beam blocker. However, this will be just equivalent to the hypothesis of the de Broglie-Bohm guiding wave and quantum potential and contradicts apparently the spirit and the simplicity of Ψ -independent models satisfying Eq. 7.

4 An example

We point out that despite these apparent contradictions it is easy to create an hidden variable model satisfying all the requirements of PBR theorem. Let any state $|\Psi\rangle$ be defined at time $t = 0$ in the complete basis $|k\rangle$ of dimension N as $|\Psi\rangle = \sum_k^N \Psi_k|k\rangle$ with $\Psi_k = \Psi'_k + i\Psi''_k$. We introduce two hidden variables λ , and μ as the N dimensional real vectors $\lambda := [\lambda_1, \lambda_2, \dots, \lambda_N]$ and $\mu := [\mu_1, \mu_2, \dots, \mu_N]$. We thus write the probability $P_M(\xi, t, \Psi) = |\langle \xi|U(t)\Psi\rangle|^2$ of observing the out-

come ξ at time t as

$$\int P_M(\xi, t|\{\lambda_k, \mu_k\}_k) \prod_k^N \delta(\Psi'_k - \lambda_k)\delta(\Psi''_k - \mu_k)d\lambda_k d\mu_k \\ = P_M(\xi, t|\{\Psi'_k, \Psi''_k\}_k) = |\sum_k \langle \xi|U(t)|k\rangle \Psi_k|^2 \quad (8)$$

where $U(t)$ is the Schrodinger evolution operator. Since Ψ can be arbitrary we thus generally have in this model

$$P_M(\xi, t|\{\lambda_k, \mu_k\}_k) = |\sum_k \langle \xi|U(t)|k\rangle (\lambda_k + i\mu_k)|^2.$$

The explicit time variation is associated with the unitary evolution $U(t)$ which thus automatically includes contextual local or non local influences (coming from the beam blocker for example). We remark that this model is of course very formal and doesn’t provide a better understanding of the mechanism explaining the interaction processes. The hidden variable model we proposed is actually based on a earlier version shortly presented by Harrigan and Spekkens in ref. [2]. We completed the model by fixing the evolution probabilities and by considering the complex nature of wave function in the Dirac distribution. Furthermore, this model doesn’t yet satisfy the independence criterion if the quantum state is defined as $|\Psi\rangle_{12} = |\Psi\rangle_1 \otimes |\Psi\rangle_2$ in the Hilbert tensor product space. Indeed, the hidden variables $\lambda_{12,k}$ and $\mu_{12,k}$ defined in Eq. 8 are global variables for the system 1,2. If we write

$$|\Psi\rangle_{12} = \sum_{n,p}^{N_1, N_2} \Psi_{12;n,p} |n\rangle_1 \otimes |p\rangle_2 \\ = \sum_{n,p}^{N_1, N_2} \Psi_{1;n} \Psi_{2;p} |n\rangle_1 \otimes |p\rangle_2 \quad (9)$$

the indices k previously used become a doublet of indices n, p and the probability

$$P_M(\xi, t|\Psi_{12}) = |\sum_{n,p}^{N_1, N_2} \langle \xi|U(t)|n, p\rangle_{12} \Psi_{12;n,p}|^2$$

in Eq. 8 reads now:

$$\int P_M(\xi, t|\{\lambda_{12;n,p}, \mu_{12;n,p}\}_{n,p}) \\ \times \prod_n^{N_1} \prod_p^{N_2} \delta(\Psi'_{12;n,p} - \lambda_{12;n,p}) \\ \times \delta(\Psi''_{12;n,p} - \mu_{12;n,p}) d\lambda_{12;n,p} d\mu_{12;n,p} \\ = P_M(\xi, t|\{\Psi'_{12;n,p}, \Psi''_{12;n,p}\}_{n,p}) \quad (10)$$

which indeed doesn’t show any explicit separation of the hidden variables density of states for subsystems 1 and 2. However, in the case where Eq. 9 is valid we can alternatively

introduce new hidden variable vectors λ_1, λ_2 and μ_1, μ_2 such that $P_M(\xi, t|\Psi_{12})$ reads now:

$$\begin{aligned} & \int P_M(\xi, t|\{\lambda_{1;n}, \lambda_{2;p}, \mu_{1;n}, \mu_{2;n}\}_{n,p}) \\ & \times \prod_n^{N_1} \delta(\Psi'_{1;n} - \lambda_{1;n}) \delta(\Psi''_{1;n} - \mu_{1;n}) d\lambda_{1;n} d\mu_{1;n} \\ & \times \prod_p^{N_2} \delta(\Psi'_{2;p} - \lambda_{2;p}) \delta(\Psi''_{2;p} - \mu_{2;p}) d\lambda_{2;p} d\mu_{2;p} \\ & = P_M(\xi, t|\{\Psi'_{1;n}, \Psi'_{2;p}, \Psi''_{1;n}, \Psi''_{2;n}\}_{n,p}). \end{aligned} \quad (11)$$

Clearly here the density of probability $\varrho_{12}(\lambda_1, \lambda_2, \mu_1, \mu_2)$ can be factorized as $\varrho_1(\lambda_1, \mu_1) \cdot \varrho_2(\lambda_2, \mu_2)$ where

$$\begin{aligned} \varrho_1(\lambda_1, \mu_1) &= \prod_n^{N_1} \delta(\Psi'_{1;n} - \lambda_{1;n}) \delta(\Psi''_{1;n} - \mu_{1;n}) \\ \varrho_2(\lambda_2, \mu_2) &= \prod_p^{N_2} \delta(\Psi'_{2;p} - \lambda_{2;p}) \delta(\Psi''_{2;p} - \mu_{2;p}) \end{aligned} \quad (12)$$

Therefore, the independence criterion at the preparation (i.e. axiom 1) is here fulfilled.

Additionally, since by definition Eq. 8 and 10 are equivalent we have

$$\begin{aligned} P_M(\xi, t|\{\Psi'_{1;n}, \Psi'_{2;p}, \Psi''_{1;n}, \Psi''_{2;n}\}_{n,p}) \\ = P_M(\xi, t|\{\Psi'_{12;n,p}, \Psi''_{12;n,p}\}_{n,p}). \end{aligned} \quad (13)$$

Moreover, since $\Psi_{1;n}$ and $\Psi_{2;n}$ can have any complex values the following relation holds for any value of the hidden variables:

$$\begin{aligned} P_M(\xi, t|\{\lambda_{1;n}, \lambda_{2;p}, \mu_{1;n}, \mu_{2;n}\}_{n,p}) \\ = P_M(\xi, t|\{\lambda_{12;n,p}, \mu_{12;n,p}\}_{n,p}) \end{aligned} \quad (14)$$

with $\lambda_{12;n,p} + i\mu_{12;n,p} = (\lambda_{1;n} + i\mu_{1;n})(\lambda_{2;p} + i\mu_{2;p})$. This clearly define a bijection or relation of equivalence between the hidden variables $[\lambda_{12}, \mu_{12}]$ on the one side and $[\lambda_1, \mu_1, \lambda_2, \mu_2]$ on the second side. Therefore, we showed that it is always possible to define hidden variables satisfying the 2 PBR axioms: i) statistical independence at the sources or preparation

$$\varrho_{j,k}(\lambda, \lambda') = \varrho_j(\lambda) \varrho_k(\lambda')$$

(if Eq. 9 is true) and ii) Ψ -independence at the dynamic level, i.e., satisfying Eq. 7. We point out that the example discussed in this section proves that the PBR theorem is not only formal since we explicitly proposed a hidden variable model satisfying the two requirements of PBR theorem. This model is very important since it demonstrates that the de Broglie Bohm approach is not the only viable hidden variable theory. It is interesting to observe that our model corresponds to the case discussed in point i) of section 3 while Bohm's approach corresponds to the point labeled ii) in the same section 3. Additionally, the new model is fundamentally stochastic (since

the transition probabilities $P_M(\xi|\lambda)$ have numerical values in general different from 1 or 0) while Bohm's approach is deterministic.

5 Conclusion

To conclude, we analyzed the PBR theorem and showed that beside the important independence criterion already pointed out in ref. 1 there is a second fundamental postulate associated with Ψ -independence at the dynamic level (that is our Eq. 7). We showed that by abandoning this prerequisite the PBR conclusion collapses. We also analyzed the nature of those models satisfying Eq. 7 and showed that despite their classical motivations they also possess counter intuitive features when compared for example to de Broglie Bohm model. We finally constructed an explicit model satisfying the PBR axioms. More studies would be necessary to understand the physical meaning of such hidden variable models.

Submitted on August 20, 2012 / Accepted on August 22, 2012

References

1. Pusey M.F., Barrett J., Rudolph T. On the reality of the quantum state. *Nature Physics*, 2012, v. 8, 478–478.
2. Harrigan N., Spekkens R. W. Einstein, incompleteness, and the epistemic view of quantum states. *Foundations of Physics*, 2010, v.40, 125–157.
3. de Broglie L. La mécanique ondulatoire et la structure atomique de la matière et du rayonnement. *Journal de Physique et le Radium*, 1927, v. 8, 225–241.
4. We point out that we will not save the naive interpretation of PBR's theorem by using the "implicit" notation $\Lambda = (\lambda, \Psi)$ and by writing $\int P_M(\xi|\Lambda) \varrho(\Lambda) d\Lambda$ in order to hide the Ψ dependence $P_M(\xi|\Lambda) := P_M(\xi|\lambda, \Psi)$. Indeed, at the end of the day we have to compare different Ψ states and the explicit notation becomes necessary. The importance of Eq. 7 for PBR's result can therefore not be avoided.
5. See also A. Drezet, arXiv:1203.2475 (12 March 2012) for a detailed discussion of PBR's theorem.