Strain Energy Density in the Elastodynamics of the Spacetime Continuum and the Electromagnetic Field

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We investigate the strain energy density of the spacetime continuum in the Elastodynamics of the Spacetime Continuum by applying continuum mechanical results to strained spacetime. The strain energy density is a scalar. We find that it is separated into two terms: the first one expresses the dilatation energy density (the "mass" longitudinal term) while the second one expresses the distortion energy density (the "massless" transverse term). The quadratic structure of the energy relation of Special Relativity is found to be present in the theory. In addition, we find that the kinetic energy *pc* is carried by the distortion part of the deformation, while the dilatation part carries only the rest-mass energy. The strain energy density of the electromagnetic energy-momentum stress tensor is calculated. The dilatation energy density (the rest-mass energy density of the photon) is found to be 0 as expected. The transverse distortion energy density is found to include a longitudinal electromagnetic energy flux term, from the Poynting vector, that is massless as it is due to distortion, not dilatation, of the spacetime continuum. However, because this energy flux is along the direction of propagation (i.e. longitudinal), it gives rise to the particle aspect of the electromagnetic field, the photon.

1 Introduction

The Elastodynamics of the Spacetime Continuum (*STCED*) is based on the application of a continuum mechanical approach to the analysis of the spacetime continuum [1–3]. The applied stresses from the energy-momentum stress tensor result in strains in, and the deformation of, the spacetime continuum (*STC*). In this paper, we explore the resulting strain energy per unit volume, that is the strain energy density, resulting from the Elastodynamics of the Spacetime Continuum. We then calculate the strain energy density of the electromagnetic field from the electromagnetic energy-momentum stress tensor.

2 Strain energy density of the spacetime continuum

The strain energy density of the spacetime continuum is a scalar given by [4, see p. 51]

$$\mathcal{E} = \frac{1}{2}\, T^{\alpha\beta}\varepsilon_{\alpha\beta} \tag{1}$$

where $\varepsilon_{\alpha\beta}$ is the strain tensor and $T^{\alpha\beta}$ is the energy-momentum stress tensor. Introducing the strain and stress deviators from (12) and (15) respectively from Millette [2], this equation becomes

$$\mathcal{E} = \frac{1}{2}\left(t^{\alpha\beta} + t g^{\alpha\beta}\right)\left(e_{\alpha\beta} + e g_{\alpha\beta}\right). \tag{2}$$

Multiplying and using relations $e^{\alpha}{}_{\alpha} = 0$ and $t^{\alpha}{}_{\alpha} = 0$ from the definition of the strain and stress deviators, we obtain

$$\mathcal{E} = \frac{1}{2}\left(4te + t^{\alpha\beta}e_{\alpha\beta}\right). \tag{3}$$

Using (11) from [2] to express the stresses in terms of the strains, this expression becomes

$$\mathcal{E} = \frac{1}{2}\,\kappa_0\varepsilon^2 + \mu_0 e^{\alpha\beta}e_{\alpha\beta} \tag{4}$$

where the Lamé elastic constant of the spacetime continuum μ_0 is the shear modulus (the resistance of the continuum to *distortions*) and κ_0 is the bulk modulus (the resistance of the continuum to *dilatations*). Alternatively, again using (11) from [2] to express the strains in terms of the stresses, this expression can be written as

$$\mathcal{E} = \frac{1}{2\kappa_0}\, t^2 + \frac{1}{4\mu_0}\, t^{\alpha\beta}t_{\alpha\beta}. \tag{5}$$

3 Physical interpretation of the strain energy density

The strain energy density is separated into two terms: the first one expresses the dilatation energy density (the "mass" longitudinal term) while the second one expresses the distortion energy density (the "massless" transverse term):

$$\mathcal{E} = \mathcal{E}_{\parallel} + \mathcal{E}_{\perp} \tag{6}$$

where

$$\mathcal{E}_{\parallel} = \frac{1}{2}\,\kappa_0\varepsilon^2 \equiv \frac{1}{2\kappa_0}\, t^2 \tag{7}$$

and

$$\mathcal{E}_{\perp} = \mu_0 e^{\alpha\beta}e_{\alpha\beta} \equiv \frac{1}{4\mu_0}\, t^{\alpha\beta}t_{\alpha\beta}. \tag{8}$$

Using (10) from [2] into (7), we obtain

$$\mathcal{E}_{\parallel} = \frac{1}{32\kappa_0}\left[\rho c^2\right]^2. \tag{9}$$

The rest-mass energy density divided by the bulk modulus κ_0, and the transverse energy density divided by the shear modulus μ_0, have dimensions of energy density as expected.

Multiplying (5) by $32\kappa_0$ and using (9), we obtain

$$32\,\kappa_0\,\mathcal{E} = \rho^2 c^4 + 8\,\frac{\kappa_0}{\mu_0}\,t^{\alpha\beta}t_{\alpha\beta}. \tag{10}$$

Noting that $t^{\alpha\beta}t_{\alpha\beta}$ is quadratic in structure, we see that this equation is similar to the energy relation of Special Relativity [5, see p. 51] for energy density

$$\hat{E}^2 = \rho^2 c^4 + \hat{p}^{\,2} c^2 \tag{11}$$

where $\hat{E}$ is the total energy density and $\hat{p}$ the momentum density.

The quadratic structure of the energy relation of Special Relativity is thus found to be present in the Elastodynamics of the Spacetime Continuum. Equations (10) and (11) also imply that the kinetic energy pc is carried by the distortion part of the deformation, while the dilatation part carries only the rest mass energy.

This observation is in agreement with photons which are massless ($\mathcal{E}_\parallel = 0$), as will be shown in the next section, but still carry kinetic energy in the transverse electromagnetic wave distortions ($\mathcal{E}_\perp = t^{\alpha\beta}t_{\alpha\beta}/4\mu_0$).

4 Electromagnetic strain energy density

The strain energy density of the electromagnetic energy-momentum stress tensor is calculated. Note that Rationalized MKSA or SI (Système International) units are used in this paper as noted previously in [3]. In addition, the electromagnetic permittivity of free space ϵ_{em} and the electromagnetic permeability of free space μ_{em} are written with "*em*" subscripts as the "0" subscripts are used in the spacetime constants. This allows us to differentiate between μ_{em} and μ_0.

Starting from the symmetric electromagnetic stress tensor [6, see pp. 64–66]

$$\Theta^{\mu\nu} = \frac{1}{\mu_{em}}\left(F^{\mu}{}_{\alpha}F^{\alpha\nu} + \frac{1}{4}\,g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \equiv \sigma^{\mu\nu}, \tag{12}$$

with $g^{\mu\nu} = \eta^{\mu\nu}$ of signature (+ - - -), and the field-strength tensor components [6, see p. 43]

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \tag{13}$$

and

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}, \tag{14}$$

we obtain [6, see p. 66] [7, see p. 141],

$$\begin{aligned} \sigma^{00} &= \tfrac{1}{2}\left(\epsilon_{em}E^2 + \tfrac{1}{\mu_{em}}B^2\right) = \tfrac{1}{2}\epsilon_{em}\left(E^2 + c^2B^2\right) \\ \sigma^{0j} &= \sigma^{j0} = \tfrac{1}{c\mu_{em}}\,(E\times B)^j = \epsilon_{em}c\,(E\times B)^j = \tfrac{1}{c}S^{\,j} \\ \sigma^{jk} &= -\left(\epsilon_{em}E^jE^k + \tfrac{1}{\mu_{em}}B^jB^k\right) + \tfrac{1}{2}\delta^{jk}\left(\epsilon_{em}E^2 + \tfrac{1}{\mu_{em}}B^2\right) \\ &= -\epsilon_{em}\left[\left(E^jE^k + c^2B^jB^k\right) - \tfrac{1}{2}\delta^{jk}\left(E^2 + c^2B^2\right)\right] \end{aligned} \tag{15}$$

where $S^{\,j}$ is the Poynting vector, and where we use the notation $\sigma^{\mu\nu} \equiv \Theta^{\mu\nu}$ as a generalization of the σ^{ij} Maxwell stress tensor notation. Hence the electromagnetic stress tensor is given by [6, see p. 66]:

$$\sigma^{\mu\nu} = \begin{pmatrix} \tfrac{1}{2}\epsilon_{em}\left(E^2+c^2B^2\right) & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix}, \tag{16}$$

where σ^{ij} is the Maxwell stress tensor. Using the relation $\sigma_{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu}\sigma^{\mu\nu}$ to lower the indices of $\sigma^{\mu\nu}$, we obtain

$$\sigma_{\mu\nu} = \begin{pmatrix} \tfrac{1}{2}\epsilon_{em}\left(E^2+c^2B^2\right) & -S_x/c & -S_y/c & -S_z/c \\ -S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ -S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ -S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix}. \tag{17}$$

4.1 Calculation of the longitudinal (mass) term

The mass term is calculated from (7) and (17) of [2]:

$$\mathcal{E}_\parallel = \frac{1}{2\kappa_0}\,t^2 = \frac{1}{32\kappa_0}\,(\sigma^\alpha{}_\alpha)^2. \tag{18}$$

The term $\sigma^\alpha{}_\alpha$ is calculated from:

$$\begin{aligned} \sigma^\alpha{}_\alpha &= \eta_{\alpha\beta}\sigma^{\alpha\beta} \\ &= \eta_{\alpha 0}\sigma^{\alpha 0} + \eta_{\alpha 1}\sigma^{\alpha 1} + \eta_{\alpha 2}\sigma^{\alpha 2} + \eta_{\alpha 3}\sigma^{\alpha 3} \\ &= \eta_{00}\sigma^{00} + \eta_{11}\sigma^{11} + \eta_{22}\sigma^{22} + \eta_{33}\sigma^{33}. \end{aligned} \tag{19}$$

Substituting from (16) and the metric $\eta^{\mu\nu}$ of signature (+ - - -), we obtain:

$$\sigma^\alpha{}_\alpha = \frac{1}{2}\,\epsilon_{em}\left(E^2 + c^2B^2\right) + \sigma_{xx} + \sigma_{yy} + \sigma_{zz}. \tag{20}$$

Substituting from (15), this expands to:

$$\begin{aligned} \sigma^\alpha{}_\alpha &= \tfrac{1}{2}\,\epsilon_{em}\left(E^2 + c^2B^2\right) + \epsilon_{em}\left({E_x}^2 + c^2{B_x}^2\right) + \\ &+ \epsilon_{em}\left({E_y}^2 + c^2{B_y}^2\right) + \epsilon_{em}\left({E_z}^2 + c^2{B_z}^2\right) - \\ &- \tfrac{3}{2}\,\epsilon_{em}\left(E^2 + c^2B^2\right) \end{aligned} \tag{21}$$

and further,

$$\sigma^{\alpha}{}_{\alpha} = \tfrac{1}{2}\,\epsilon_{em}\left(E^2 + c^2B^2\right) + \epsilon_{em}\left(E^2 + c^2B^2\right) - \\ -\tfrac{3}{2}\,\epsilon_{em}\left(E^2 + c^2B^2\right). \tag{22}$$

Hence

$$\sigma^{\alpha}{}_{\alpha} = 0 \tag{23}$$

and, substituting into (18),

$$\mathcal{E}_{\|} = 0 \tag{24}$$

as expected [6, see pp. 64–66]. This derivation thus shows that the rest-mass energy density of the photon is 0.

4.2 Calculation of the transverse (massless) term

The transverse term is calculated from (8), viz.

$$\mathcal{E}_{\perp} = \frac{1}{4\mu_0}\, t^{\alpha\beta} t_{\alpha\beta}. \tag{25}$$

Given that $t = \frac{1}{4}\sigma^{\alpha}{}_{\alpha} = 0$, then $t^{\alpha\beta} = \sigma^{\alpha\beta}$ and the terms $\sigma^{\alpha\beta}\sigma_{\alpha\beta}$ are calculated from the components of the electromagnetic stress tensors of (16) and (17). Substituting for the diagonal elements and making use of the symmetry of the Poynting component terms and of the Maxwell stress tensor terms from (16) and (17), this expands to:

$$\begin{aligned} \sigma^{\alpha\beta}\sigma_{\alpha\beta} &= \tfrac{1}{4}\,\epsilon_{em}^2\left(E^2 + c^2B^2\right)^2 + \\ &+\epsilon_{em}^2\left[\left(E_xE_x + c^2B_xB_x\right) - \tfrac{1}{2}\left(E^2 + c^2B^2\right)\right]^2 + \\ &+\epsilon_{em}^2\left[\left(E_yE_y + c^2B_yB_y\right) - \tfrac{1}{2}\left(E^2 + c^2B^2\right)\right]^2 + \\ &+\epsilon_{em}^2\left[\left(E_zE_z + c^2B_zB_z\right) - \tfrac{1}{2}\left(E^2 + c^2B^2\right)\right]^2 - \\ &-2\left(S_x/c\right)^2 - 2\left(S_y/c\right)^2 - 2\left(S_z/c\right)^2 + \\ &+2\left(\sigma_{xy}\right)^2 + 2\left(\sigma_{yz}\right)^2 + 2\left(\sigma_{zx}\right)^2. \end{aligned} \tag{26}$$

The E-B terms expand to:

$$\begin{aligned} \text{EBterms} &= \epsilon_{em}^2\left[\tfrac{1}{4}\left(E^2 + c^2B^2\right)^2 + \right. \\ &+\left(E_x{}^2 + c^2B_x{}^2\right)^2 - \left(E_x{}^2 + c^2B_x{}^2\right)\left(E^2 + c^2B^2\right) + \\ &+\left(E_y{}^2 + c^2B_y{}^2\right)^2 - \left(E_y{}^2 + c^2B_y{}^2\right)\left(E^2 + c^2B^2\right) + \\ &+\left(E_z{}^2 + c^2B_z{}^2\right)^2 - \left(E_z{}^2 + c^2B_z{}^2\right)\left(E^2 + c^2B^2\right) + \\ &\left. +\tfrac{3}{4}\left(E^2 + c^2B^2\right)^2\right]. \end{aligned} \tag{27}$$

Simplifying,

$$\begin{aligned} \text{EBterms} &= \epsilon_{em}^2\left[\left(E^2 + c^2B^2\right)^2 - \left(E_x{}^2 + c^2B_x{}^2 + \right.\right. \\ &\left. +E_y{}^2 + c^2B_y{}^2 + E_z{}^2 + c^2B_z{}^2\right)\left(E^2 + c^2B^2\right) + \\ &+\left(E_x{}^2 + c^2B_x{}^2\right)^2 + \left(E_y{}^2 + c^2B_y{}^2\right)^2 + \\ &\left. +\left(E_z{}^2 + c^2B_z{}^2\right)^2\right] \end{aligned} \tag{28}$$

which gives

$$\begin{aligned} \text{EBterms} &= \epsilon_{em}^2\left[\left(E^2 + c^2B^2\right)^2 - \left(E^2 + c^2B^2\right)^2 + \right. \\ &+\left(E_x{}^2 + c^2B_x{}^2\right)^2 + \left(E_y{}^2 + c^2B_y{}^2\right)^2 + \\ &\left. +\left(E_z{}^2 + c^2B_z{}^2\right)^2\right] \end{aligned} \tag{29}$$

and finally

$$\begin{aligned} \text{EBterms} &= \epsilon_{em}^2\left[\left(E_x{}^4 + E_y{}^4 + E_z{}^4\right) + \right. \\ &+c^4\left(B_x{}^4 + B_y{}^4 + B_z{}^4\right) + \\ &\left. +2c^2\left(E_x{}^2B_x{}^2 + E_y{}^2B_y{}^2 + E_z{}^2B_z{}^2\right)\right]. \end{aligned} \tag{30}$$

Including the E-B terms in (26), substituting from (15), expanding the Poynting vector and rearranging, we obtain

$$\begin{aligned} \sigma^{\alpha\beta}\sigma_{\alpha\beta} &= \epsilon_{em}^2\left[\left(E_x{}^4 + E_y{}^4 + E_z{}^4\right) + c^4\left(B_x{}^4 + B_y{}^4 + \right.\right. \\ &\left.\left. +B_z{}^4\right) + 2c^2\left(E_x{}^2B_x{}^2 + E_y{}^2B_y{}^2 + E_z{}^2B_z{}^2\right)\right] - \\ &-2\epsilon_{em}^2c^2\left[\left(E_yB_z - E_zB_y\right)^2 + \left(-E_xB_z + E_zB_x\right)^2 + \right. \\ &\left. +\left(E_xB_y - E_yB_x\right)^2\right] + 2\epsilon_{em}^2\left[\left(E_xE_y + c^2B_xB_y\right)^2 + \right. \\ &\left. +\left(E_yE_z + c^2B_yB_z\right)^2 + \left(E_zE_x + c^2B_zB_x\right)^2\right]. \end{aligned} \tag{31}$$

Expanding the quadratic expressions,

$$\begin{aligned} \sigma^{\alpha\beta}\sigma_{\alpha\beta} &= \epsilon_{em}^2\left[\left(E_x{}^4 + E_y{}^4 + E_z{}^4\right) + c^4\left(B_x{}^4 + B_y{}^4 + \right.\right. \\ &\left.\left. +B_z{}^4\right) + 2c^2\left(E_x{}^2B_x{}^2 + E_y{}^2B_y{}^2 + E_z{}^2B_z{}^2\right)\right] - \\ &-2\epsilon_{em}^2c^2\left[E_x{}^2B_y{}^2 + E_y{}^2B_z{}^2 + E_z{}^2B_x{}^2 + B_x{}^2E_y{}^2 + \right. \\ &+B_y{}^2E_z{}^2 + B_z{}^2E_x{}^2 - 2\left(E_xE_yB_xB_y + E_yE_zB_yB_z + \right. \\ &\left.\left. +E_zE_xB_zB_x\right)\right] + 2\epsilon_{em}^2\left[\left(E_x{}^2E_y{}^2 + E_y{}^2E_z{}^2 + \right.\right. \end{aligned} \tag{32}$$

$$+E_z{}^2E_x{}^2\Big) + 2c^2\left(E_xE_yB_xB_y + E_yE_zB_yB_z + E_zE_xB_zB_x\right) + c^4\left(B_x{}^2B_y{}^2 + B_y{}^2B_z{}^2 + B_z{}^2B_x{}^2\right)\Big]$$

Grouping the terms in powers of c together,

$$\begin{aligned}\frac{1}{\epsilon_{em}^2}\,\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \Big[&\left(E_x{}^4 + E_y{}^4 + E_z{}^4\right) + 2\left(E_x{}^2E_y{}^2 + E_y{}^2E_z{}^2 + E_z{}^2E_x{}^2\right)\Big] + 2c^2\Big[\left(E_x{}^2B_x{}^2 + E_y{}^2B_y{}^2 + E_z{}^2B_z{}^2\right) - \\ &-\left(E_x{}^2B_y{}^2 + E_y{}^2B_z{}^2 + E_z{}^2B_x{}^2 + B_x{}^2E_y{}^2 + B_y{}^2E_z{}^2 + B_z{}^2E_x{}^2\right) + 4\left(E_xE_yB_xB_y + E_yE_zB_yB_z + E_zE_xB_zB_x\right)\Big] + \\ &+ c^4\Big[\left(B_x{}^4 + B_y{}^4 + B_z{}^4\right) + 2\left(B_x{}^2B_y{}^2 + B_y{}^2B_z{}^2 + B_z{}^2B_x{}^2\right)\Big].\end{aligned} \tag{33}$$

Simplifying,

$$\begin{aligned}\frac{1}{\epsilon_{em}^2}\,\sigma^{\alpha\beta}\sigma_{\alpha\beta} = &\left(E_x{}^2 + E_y{}^2 + E_z{}^2\right)^2 + 2c^2\left(E_x{}^2 + E_y{}^2 + E_z{}^2\right)\left(B_x{}^2 + B_y{}^2 + B_z{}^2\right) - \\ &-2c^2\Big[2\Big(E_x{}^2B_y{}^2 + E_y{}^2B_z{}^2 + E_z{}^2B_x{}^2 + B_x{}^2E_y{}^2 + B_y{}^2E_z{}^2 + B_z{}^2E_x{}^2\Big) - 4\left(E_xE_yB_xB_y + E_yE_zB_yB_z + E_zE_xB_zB_x\right)\Big] + c^4\left(B_x{}^2 + B_y{}^2 + B_z{}^2\right)^2\end{aligned} \tag{34}$$

which is further simplified to

$$\begin{aligned}\frac{1}{\epsilon_{em}^2}\,\sigma^{\alpha\beta}\sigma_{\alpha\beta} = &\left(E^4 + 2c^2E^2B^2 + c^4B^4\right) - \\ &-4c^2\Big[\left(E_yB_z - B_yE_z\right)^2 + (E_zB_x - B_zE_x)^2 + \left(E_xB_y - B_xE_y\right)^2\Big].\end{aligned} \tag{35}$$

Making use of the definition of the Poynting vector from (15), we obtain

$$\begin{aligned}\sigma^{\alpha\beta}\sigma_{\alpha\beta} = &\,\epsilon_{em}^2\left(E^2 + c^2B^2\right)^2 - \\ &-4\epsilon_{em}^2c^2\left[(E\times B)_x{}^2 + (E\times B)_y{}^2 + (E\times B)_z{}^2\right]\end{aligned} \tag{36}$$

and finally

$$\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \epsilon_{em}^2\left(E^2 + c^2B^2\right)^2 - \frac{4}{c^2}\left(S_x{}^2 + S_y{}^2 + S_z{}^2\right). \tag{37}$$

Substituting in (25), the transverse term becomes

$$\mathcal{E}_\perp = \frac{1}{4\mu_0}\left[\epsilon_{em}^2\left(E^2 + c^2B^2\right)^2 - \frac{4}{c^2}\,S^2\right] \tag{38}$$

or

$$\mathcal{E}_\perp = \frac{1}{\mu_0}\left[U_{em}{}^2 - \frac{1}{c^2}\,S^2\right] \tag{39}$$

where $U_{em} = \frac{1}{2}\,\epsilon_{em}\,(E^2 + c^2B^2)$ is the electromagnetic field energy density.

4.3 Electromagnetic field strain energy density and the photon

$\boldsymbol{S}$ is the electromagnetic energy flux along the direction of propagation [6, see p. 62]. As noted by Feynman [8, see pp. 27-1–2], local conservation of the electromagnetic field energy can be written as

$$-\frac{\partial U_{em}}{\partial t} = \nabla\cdot\boldsymbol{S}, \tag{40}$$

where the term $\boldsymbol{E}\cdot\boldsymbol{j}$ representing the work done on the matter inside the volume is 0 in the absence of charges (due to the absence of mass [3]). By analogy with the current density four-vector $j^\nu = (c\varrho, \boldsymbol{j})$, where ϱ is the charge density, and $\boldsymbol{j}$ is the current density vector, which obeys a similar conservation relation, we define the Poynting four-vector

$$S^\nu = (cU_{em}, \boldsymbol{S}), \tag{41}$$

where U_{em} is the electromagnetic field energy density, and $\boldsymbol{S}$ is the Poynting vector. Furthermore, as per (40), S^ν satisfies

$$\partial_\nu S^\nu = 0. \tag{42}$$

Using definition (41) in (39), that equation becomes

$$\mathcal{E}_\perp = \frac{1}{\mu_0c^2}\,S_\nu S^\nu. \tag{43}$$

The indefiniteness of the location of the field energy referred to by Feynman [8, see p. 27-6] is thus resolved: the electromagnetic field energy resides in the distortions (transverse displacements) of the spacetime continuum.

Hence the invariant electromagnetic strain energy density is given by

$$\mathcal{E} = \frac{1}{\mu_0c^2}\,S_\nu S^\nu \tag{44}$$

where we have used $\rho = 0$ as per (23). This confirms that S^ν as defined in (41) is a four-vector.

It is surprising that a longitudinal energy flow term is part of the transverse strain energy density i.e. S^2/μ_0c^2 in (39). We note that this term arises from the time-space components of (16) and (17) and can be seen to correspond to the transverse displacements along the *time-space* planes which are folded along the direction of propagation in 3-space as the Poynting vector. The electromagnetic field energy density term $U_{em}{}^2/\mu_0$ and the electromagnetic field energy flux term S^2/μ_0c^2 are thus combined into the transverse strain energy density. The negative sign arises from the signature (+ - - -) of the metric tensor $\eta^{\mu\nu}$.

This longitudinal electromagnetic energy flux is massless as it is due to distortion, not dilatation, of the spacetime continuum. However, because this energy flux is along the direction of propagation (i.e. longitudinal), it gives rise to the particle aspect of the electromagnetic field, the photon. As shown in [9, see pp. 174-5] [10, see p. 58], in the quantum theory of electromagnetic radiation, an intensity operator derived from the Poynting vector has, as expectation value, photons in the direction of propagation.

This implies that the $(pc)^2$ term of the energy relation of Special Relativity needs to be separated into transverse and longitudinal massless terms as follows:

$$\hat{E}^2 = \underbrace{\rho^2 c^4}_{\mathcal{E}_\parallel} + \underbrace{\hat{p}_\parallel^2 c^2 + \hat{p}_\perp^2 c^2}_{massless\ \mathcal{E}_\perp} \tag{45}$$

where $\hat{p}_\parallel$ is the massless longitudinal momentum density. Equation (39) shows that the electromagnetic field energy density term ${U_{em}}^2/\mu_0$ is reduced by the electromagnetic field energy flux term $S^2/\mu_0 c^2$ in the transverse strain energy density, due to photons propagating in the longitudinal direction. Thus the kinetic energy is carried by the distortion part of the deformation, while the dilatation part carries only the rest-mass energy, which in this case is 0.

As shown in (9), (10) and (11), the constant of proportionality to transform energy density squared ($\hat{E}^2$) into strain energy density ($\mathcal{E}$) is $1/(32\kappa_0)$:

$$\mathcal{E}_\parallel = \frac{1}{32\kappa_0}\left[\rho c^2\right]^2 \tag{46}$$

$$\mathcal{E} = \frac{1}{32\kappa_0}\hat{E}^2 \tag{47}$$

$$\mathcal{E}_\perp = \frac{1}{32\kappa_0}\left[\hat{p}_\parallel^2 c^2 + \hat{p}_\perp^2 c^2\right] = \frac{1}{4\mu_0} t^{\alpha\beta} t_{\alpha\beta}. \tag{48}$$

Substituting (39) into (48), we obtain

$$\mathcal{E}_\perp = \frac{1}{32\kappa_0}\left[\hat{p}_\parallel^2 c^2 + \hat{p}_\perp^2 c^2\right] = \frac{1}{\mu_0}\left[{U_{em}}^2 - \frac{1}{c^2} S^2\right] \tag{49}$$

and

$$\hat{p}_\parallel^2 c^2 + \hat{p}_\perp^2 c^2 = \frac{32\kappa_0}{\mu_0}\left[{U_{em}}^2 - \frac{1}{c^2} S^2\right] \tag{50}$$

This suggests that

$$\mu_0 = 32\kappa_0, \tag{51}$$

to obtain the relation

$$\hat{p}_\parallel^2 c^2 + \hat{p}_\perp^2 c^2 = {U_{em}}^2 - \frac{1}{c^2} S^2. \tag{52}$$

5 Discussion and conclusion

In this paper, we have analyzed the strain energy density of the spacetime continuum in *STCED* and evaluated it for the electromagnetic stress tensor. We have found that the strain energy density is separated into two terms: the first one expresses the dilatation energy density (the "mass" longitudinal term) while the second one expresses the distortion energy density (the "massless" transverse term). We have found that the quadratic structure of the energy relation of Special Relativity is present in the strain energy density of the Elastodynamics of the Spacetime Continuum. We have also found that the kinetic energy *pc* is carried by the distortion part of the deformation, while the dilatation part carries only the rest mass energy.

We have calculated the strain energy density of the electromagnetic energy-momentum stress tensor. We have found that the dilatation longitudinal (mass) term of the strain energy density and hence the rest-mass energy density of the photon is 0. We have found that the distortion transverse (massless) term of the strain energy density is a combination of the electromagnetic field energy density term ${U_{em}}^2/\mu_0$ and the electromagnetic field energy flux term $S^2/\mu_0 c^2$, calculated from the Poynting vector. This longitudinal electromagnetic energy flux is massless as it is due to distortion, not dilatation, of the spacetime continuum. However, because this energy flux is along the direction of propagation (i.e. longitudinal), it gives rise to the particle aspect of the electromagnetic field, the photon.

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