The Gravitational Field: A New Approach

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In this paper, we consider the Einstein field equations with the cosmological term. If we assume that this term is slightly varying, it induces a vacuum background field filling the space. In this case, inspection shows that the gravitational field is no longer represented by a pseudo-tensor, but appears on the right hand side of the field equations as a true tensor together with the bare mass tensor thus restoring the same conservation condition as obeyed by the Einstein tensor.

Introduction

Soon after his theory of General Relativity was published in 1916, Einstein rapidly turned to the unifying of the gravitational field with electromagnetism (which at that time was considered as the second fundamental field).

The quest for such an universal scheme ended in 1955 with the Einstein-Schrödinger theory (see for example [1]) definitely abandoned since as the quantum field theories gained the increasing successes and have been long substantiated by numerous experimental confirmations.

Basically, the unified principle adopted by the successive authors (Kaluza-Klein, Weyl, Eddington, et al.) relied either on extra dimensions, or on an extension of the Riemannian theory with additional space-time curvatures introduced to yield the electromagnetic field characteristics, and where the stress-energy tensor regarded as provisional, will be eventually absent [2, 3, 4].

Total geometrization of matter and electromagnetism was anyhow the original focus.

To understand this long period of research, one should remember that Einstein always claimed that the energy-momentum tensor \((s)\) which can appear in the right hand side of his field equations, was “clumsy”; in short, he considered this form as an unsatisfactory solution which had to fit differently in his equations.

Einstein’s argument is actually strongly supported by the following fact: while his tensor exhibits a conceptually conserved property, any corresponding stress energy-tensor does not, which leaves the theory with a major inconsistency.

When pure matter is the source, the problem has been “cured” by introducing the so-called “pseudo-tensor” that “conveniently” describes the gravitational field of this mass so that the four-momentum of both matter and its gravity field is conserved.

Unfortunately by essence this pseudo-tensor cannot appear in the field equations, and so the obvious physical defect emphasized by Einstein, still remains to-day as a stumbling block.

In this paper, we tackle this problems by proceeding as follows: in contrast to the previous theories, the energy-momentum tensor of the source is here strengthened, although we restrict our study to neutral massive flow.

In this respect, it is shown that the gravitational field of a massive body is no longer described by a pseudo-tensor, but appears as a true tensor in the field equations as it should be, in order to balance the conceptually conserved property of the Einstein tensor.

To achieve this goal we do:

- We first formulate the field equations with a massive source in density notation;
- We write the conservation law for the Einstein tensor density derived from the Bianchi identities, which cannot apply to the energy-momentum tensor density as a source;
- We then include a variable term that supersedes the so-called cosmological term \(\Lambda g_{ab}\) in the field equations, still complying with the conservation property of the Einstein tensor density in GR;
- Under this latter assumption, we will then formally show that the gravity field of a massive source is no longer described by a vanishing pseudo tensor but it reduces to a true tensor describing a persistent vacuum background field resulting from the existence of the variable term.

1 The field equations in General Relativity

1.1 The tensor representation

In the General Theory of Relativity (GR), it is well known that by varying the action

\[ S = L_E d^4x, \]

where the Lagrangian density is given by

\[ L_E = \sqrt{-g} G^{ab} \left( \epsilon_{[ab]} \left[ \epsilon_{da} \epsilon_{b]c} \right] \right), \quad (1.1) \]
\[ g = \det \left| g_{ab} \right| \quad (1.2) \]

one infers the symmetric Einstein tensor

\[ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \quad (1.3) \]
where
\[ R_{bc} = \partial_a \left[ \frac{\alpha}{\epsilon} \right] - \partial_c \left[ \frac{\alpha}{\epsilon} \right] + \frac{\delta}{\epsilon} \left[ \frac{\alpha}{\epsilon} \right] - \frac{\delta}{\epsilon} \left[ \frac{\alpha}{\epsilon} \right] \] (1.4)
is the Ricci tensor with its contraction \( R \), the curvature scalar, while \( \{ \alpha \} \) denote the Christoffel Symbols of the second kind.

The 10 source free field equations are
\[ G_{ab} = 0. \] (1.5)

The second rank Einstein tensor \( G_{ab} \) is symmetric and is only function of the metric tensor components \( g_{ab} \) and their first and second order derivatives.

The relation
\[ \nabla_a \dot{g}_{b}^{a} = 0 \] (1.6)
is the conservation identities provided that the tensor \( G_{ab} \) has the form [5]
\[ G_{ab} = k \left[ R_{ab} - \frac{1}{2} g_{ab} (R - 2\Lambda) \right] \] (1.7)
\( k \) is a constant, which is here taken 1, is usually named cosmological constant.

When a source is present, the field equations become
\[ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R - g_{ab} \Lambda = \kappa T_{ab}, \] (1.8)
where \( T_{ab} \) is the energy-momentum tensor of the source.

1.2 The tensor density representation

We first set
\[ g^{ab} = \sqrt{-g} g^{ab} \] (1.9)
and the Einstein tensor density is
\[ G^{ab} = \sqrt{-g} G_{ab}, \quad G_{ab} = \sqrt{-g} G_{ab}^{\epsilon}, \] (1.10)
\[ R^{ab} = \sqrt{-g} R_{ab}. \] (1.11)

In density notations, the field equations with the source (1.8) will read
\[ G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R - g^{ab} \kappa = \kappa T^{ab}. \] (1.12)

Here in place of the constant cosmological term \( \Lambda \) which should be here represented by \( \Lambda \sqrt{-g} \), we have introduced a scalar density denoted as
\[ \kappa = \Xi \sqrt{-g}. \] (1.13)

Unlike \( \Lambda \), the scalar \( \Xi \) is slightly variable and represents the Lagrangian characterizing a specific vacuum background field as will be shown below.

2 The conservation identities

2.1 Tensor version for the Einstein tensor

From the Bianchi identities applied to the Riemann tensor
\[ R_{\gamma \delta ; \mu} + R_{\gamma \delta ; \nu} + R_{\gamma \delta ; \sigma} = 0 \] (2.1)
we infer the conservation conditions which apply to the Einstein tensor without \( \Xi \), and hereinafter denoted by
\[ \dot{g}^{ab} = R^{ab} - \frac{1}{2} g^{ab} R. \] (2.2)

The Einstein tensor thus satisfies intrinsically the conservation law:
\[ \nabla_a \dot{g}^{ab} = 0, \] (2.3)

2.2 Tensor density version for the Einstein tensor

In the same way, we start with the Einstein tensor density without the cosmological term
\[ \dot{g}^{ab} = R^{ab} - \frac{1}{2} g^{ab} R. \] (2.4)

With (2.3), let us write down
\[ \nabla_a \dot{g}^{ab} = \frac{\partial_a \dot{g}^{ab}}{\sqrt{-g}} \partial_b g_{ab} = 0 \] (2.5)
using \( dg_{ab} = -g_{ab} \dot{g}_{ac} d\dot{g}^{bc} \) and \( d\dot{g}^{ab} = -g^{ab} g^{cd} d\dot{g}_{bc} \) the formula (2.5) can be also written as
\[ \partial_a \dot{g}^{ab} = \frac{1}{2} \dot{g}^{ab} \partial_b g_{ab} = 0. \] (2.6)

The latter equation is the conservation condition for \( \dot{g}^{ab} \) which is equivalent to (2.3).

2.3 Conservation of the energy-momentum tensor

2.3.1 Problem statement

Let us consider the energy-momentum tensor for neutral matter density \( \rho \):
\[ T_{ab} = \rho u_a u_b \] (2.7)
as the right hand side of the field equations
\[ \dot{g}^{ab} = R^{ab} - \frac{1}{2} g^{ab} R = \kappa T^{ab}. \] (2.8)

The conservation condition for this tensor are written
\[ \nabla_a T_{ba}^{a} = \frac{1}{\sqrt{-g}} \partial_a T_{ba}^{a} - \frac{1}{2} T^{ac} \partial_b g_{ac} = 0 \] (2.9)
with the tensor density
\[ T^a_b = \sqrt{-g} T^a_b. \] (2.10)

However, across a given hypersurface \( dS_b \), the integral
\[ P^a = \int T^{ab} \sqrt{-g} \, dS_b \] (2.11)
is conserved only when
\[ \partial_a T^a_b = 0. \] (2.12)

From (2.6) inspection still shows that
\[ \partial_a T^a_b = \frac{1}{2} T^{cd} \partial_b g_{cd}. \] (2.13)

but here, unlike the Einstein tensor \( ^eG_{ab} \) which is conceptually conserved \((\nabla_aG^a_b = 0)\), the conditions
\[ \nabla_a T^a_b = 0 \]
or
\[ \partial_a T^a_b = 0 \]
are thus never satisfied in a general coordinates system.

Therefore, the Einstein tensor \( ^eG_{ab} \) which intrinsically obeys a conservation condition, is related with a massive tensor \( T_{ab}(\rho) \) which obviously fails to satisfy the same requirement:
\[ ^eG_{ab} = \propto T_{ab}. \] (2.14)

As a matter of fact, a correct formulation would consist of explicitly writing down the mass density with its gravity field, i.e. with a pseudo-tensor \((T_{ab})\)-field.

As is known, the name pseudo-tensor is chosen since this quantity can be transformed away by a suitable choice of coordinates.

Hence, we should write
\[ G_{ab} = \propto \left[(T_{ab})\text{matter} + (T_{ab})\text{field}\right]. \] (2.15)

This is classically interpreted by requiring that the total 4-momentum vector \( P^a \) of matter with its gravitational field
\[ P^a = \left[(T_{ab})\text{matter} + (T_{ab})\text{field}\right] \sqrt{-g} \, dS_b \] (2.16)
must be together conserved*.

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2.3.2 The gravity pseudo-tensor

In order to follow this way, Landau and Lifshitz [6] started from the unsuitable tensor equation (2.9)
\[ \nabla_k T^k_i = \frac{1}{\sqrt{-g}} \partial_i T^k_k - \frac{1}{2} T^{kl} \partial_i g_{kl} = 0. \]

They thus consider a special choice of a set of the coordinates which cancels out all first derivatives of the \( g_{ik} \) at a given 4-space-time point.

In this system, the energy-momentum tensor expression is given by
\[ T^{ik} = \frac{1}{2\kappa} \partial_i (-g)^{-1} \partial_d (-g) \left\{ g^{ik} g^{cd} - g^{ic} g^{kd} \right\}. \] (2.17)

As \( \partial_i (-g)^{-1} \) are postulated to be zero at the considered point, we may extract the factor \((-g)^{-1}\) from the derivative in the latter equation, so
\[ (-g) T^{ik} = \partial_i H^{ik} = \frac{1}{2\kappa} \partial_i \left( \partial_d H^{ked} \right). \]

The quantity
\[ H^{ik} = (-g)(T^{ik} + \Lambda^{ik}) \]
can be regarded as a “double tensor density” and is often referred to, as the “superpotential of Landau-Lifshitz” [7].

Now, in any other arbitrary system, generally
\[ \partial_i H^{ik} = (-g)(T^{ik} + \Lambda^{ik}), \]
and so, we will have to bring a small tensor correction \( \Lambda^{ik} \) (Landau-Lifshitz pseudo-tensor) which is accepted as representing the gravitational field of matter:
\[ \partial_i H^{ik} = (-g)(T^{ik} + \Lambda^{ik}). \]

This equation implies the condition
\[ \partial_k \left[ (-g)(T^{ik} + \Lambda^{ik}) \right] = 0, \] (2.19)
which is the conservation law for the classical total four-momentum vector density of both matter and gravitational field written as
\[ P^a = \int \left[ (-g)(T^{ik} + \Lambda^{ik}) \right] dS_k, \] (2.20)
(compare with (2.11)).

After a tedious calculation, the final form of the symmetric tensor \( \Lambda^{ik} \) as a function of the \( g_{ik} \), is found to be
\[ (-g) \Lambda^{ik} = \frac{1}{2\kappa} \left\{ g^{ik} g^{lm} g_{km} - g_{ik} g^{lm} g^{km} + \frac{1}{2} g^{ik} g^{lm} g_{km} - \left( g^{il} g^{km} g_{jk} + g^{kl} g^{im} g_{jm} \right) g_{lm} g^{np} g^{ij} g^{kp} + \frac{1}{8} \left( 2g^{ik} g^{lm} - g^{ik} g^{lm} \right) \left( 2g_{np} g_{qr} - g_{pq} g_{nr} \right) g^{ij} g^{kp} \right\}. \] (2.21)

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*Some authors [8] state that integrating \( \nabla_k T^k_i = 0 \) yields a conservation law for a vector \( P^a = T^{ab} K_b \) when the metric admits a Killing vector \( K_b \),\begin{align*}
P^a &= T_{ab} K_b + T^{ab} K_{ab} \quad \text{and since} \quad T^{ab} \quad \text{is symmetric, we have for the Lie derivative} \quad K_{ab} = \frac{1}{2} L_K g_{ab} = 0, \quad \text{then} \quad P^a = 0.
\end{align*}
Therefore, the Einstein field equations can be eventually written in the form:

\[ H^{t^i^j}_{\cd} = 2\kappa (-g) \left( T^{cie} + t^{cie}_{\text{field}} \right). \]  

(2.22)

Unfortunately, the quantity \( t^{cie}_{\text{field}} \) which now appears on the right hand side of the field equations as it should be, is not a true tensor.

Hence, we are once more faced with a contradiction: the left hand side of the field equations for a massive source is a true tensor, while the right hand side is not, which reveals a major inconsistency within the theory.

### 2.4 Introduction of a background field tensor

Let us now try to remove this ambiguity.

We start by writing the global energy-momentum tensor density of the massive source splitting up bare matter and pure field:

\[ T^i_j = (T^i_j)_{\text{matter}} + (t^i_j)_{\text{field}}. \]  

(2.23)

The field tensor density \((t^i_j)_{\text{field}}\) is in turn composed of two parts: gravity field + vacuum background field

\[ (t^i_j)_{\text{field}} = (t^i_j)_{\text{gravity}} + (t^i_j)_{\text{background field}} \]  

(2.24)

with

\[ \frac{\zeta}{2\kappa} g_{ab} = \frac{\Xi}{2\kappa} g_{ab}. \]  

(2.25)

According to the standard theory, we next re-formulate the field equations with a bare massive source

\[ G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R = \kappa (T^{ab})_{\text{matter}} \]  

(2.26)

under the form

\[ G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R = \kappa (T^{ab})_{\text{matter}} + g^{ab} \zeta. \]  

(2.27)

### 3 Explicitly the field equations in density notation

#### 3.1 Taking account of the Lagrangian \( \Xi \)

Reverting to (2.13), we now write for the bare matter tensor density

\[ \partial_0 (T^i_j)_{\text{matter}} = \frac{1}{2} (T^i_j)_{\text{matter}} \delta_0 g_{cd}. \]  

(3.1)

Inspection then shows that

\[ R^i_j \delta g^i = \sqrt{-g} \left[ -R^{cie} + \frac{1}{2} g^{ie} R \right] \delta g_{ie} = -\kappa (T^{cie})_{\text{matter}} \delta g_{ie}. \]  

(3.2)

Taking now into account the Lagrangian formulation for \( R^i_j \), which is

\[ R^i_j = \frac{\partial L_E}{\delta g^i} = \partial_k \frac{\partial L_E}{\delta (\partial_t g^{ik})} - \frac{\partial L_E}{\delta g^{i0}} \]  

(3.3)

we obtain

\[ -\kappa (T^{i0})_{\text{matter}} \delta g_{0} = \partial_k \frac{\partial L_E}{\delta (\partial_t g^{ik})} - \frac{\partial L_E}{\delta g^{i0}} \delta g^0 = \frac{\partial L_E}{\delta (\partial_t g^{i0})} \delta g^0 - L_E, \]

that is

\[ -\kappa (T^{i0})_{\text{matter}} \delta m g_{0} = \partial_k \left[ \frac{\partial L_E}{\delta (\partial_t g^{ik})} - \frac{\partial L_E}{\delta g^{i0}} \delta g^0 \right] = \frac{\partial L_E}{\delta (\partial_t g^{i0})} \delta g^0 - L_E, \]  

(3.4)

where \((t^i_j)_{\text{field}}\) denotes the field tensor density extracted from

\[ \frac{\partial k (t^i_j)_{\text{field}}} = \frac{1}{2\kappa} \left[ \frac{\partial L_E}{\delta (\partial_t g^{ik})} - \frac{\partial L_E}{\delta g^{i0}} \delta g^0 \right] \]  

(3.5)

so, that we have the explicit canonical form

\[ (t^i_j)_{\text{field}} = \frac{1}{2\kappa} \left[ \frac{\partial L_E}{\delta (\partial_t g^{ik})} - \frac{\partial L_E}{\delta g^{i0}} \delta g^0 \right] \]  

(3.6)

and where

\[ \partial_k (T^i_j)_{\text{matter}} = \frac{1}{2} (T^i_j)_{\text{matter}} \partial_0 g_{0} = -\partial_k (t^i_j)_{\text{field}}. \]

That is, the required conservation relation

\[ \partial_k \left[ (T^i_j)_{\text{matter}} + (t^i_j)_{\text{field}} \right] = 0. \]  

(3.7)

Then, re-instating the term \( \zeta \) according to (2.24) and (2.25), the gravitational field tensor density now reads:

\[ (t^i_j)_{\text{gravity}} = \frac{1}{2\kappa} \left[ \frac{\partial L_E}{\delta (\partial_t g^{ik})} - \frac{\partial L_E}{\delta g^{i0}} \delta g^0 \right] \]  

(3.8)

The presence of the scalar density \( \zeta \) characterizing the background field is here of central importance, as it means that \((t^i_j)_{\text{gravity}}\) can never be zero in contrast to the classical theory, and as a result, it constitutes a true tensor. Such a gravity field never completely cancels out, but far from its matter source, it sharply decreases down to the level of the background field described by the tensor density \((t^i_j)_{\text{background field}}\).

In addition, we clearly see that \( \zeta \) represents the lagrangian density characterizing the background field, thus lending support to our initial hypothesis regarding the lagrangian \( \Xi \).

In this picture, the vacuum is permanently filled with this homogeneous background energy field ensuring a smooth continuity with the gravitational field of a neighbouring mass.

#### 3.2 Classical formulation

When the term \( \Xi \) is kept constant like the cosmological term \( \Lambda \), the tensor density (3.8) reduces to

\[ (t^i_j)_{\text{pseudo}} = \frac{1}{2\kappa} \left[ \frac{\partial L_E}{\delta (\partial_t g^{ik})} - \frac{\partial L_E}{\delta g^{i0}} \delta g^0 \right] \]  

(3.9)
which is just the classical gravity pseudo-tensor density that may now vanish in a given space-time point.

In this case, expressed with the explicit form of the Lagrangian density \( L_E \) written in (1.1), the expression (3.9) becomes:

\[
(T^\mu_\mu)_{\text{pseudo-gravity}} = \frac{1}{2\kappa} \left[ \sum_I \left[ \frac{\partial}{\partial x^i} \left( g^{ij} \frac{\partial \Phi}{\partial x^j} - \Phi \frac{\partial g^{ij}}{\partial x^j} \right) - \frac{\delta}{\delta x^i} L_E \right] \right]. \tag{3.10}
\]

This is the mixed Einstein-Dirac pseudo-tensor density [9] which is not symmetric on \( k \) and \( m \), and is therefore not suitable for basing a definition of angular momentum on.

3.3 Field equations

The field equations with a massive source, which are the skew components of the energy-momentum tensor (3.17), which implicitly contains the gravity field [10]. Now, we formulate (3.18) under the equivalent form:

\[
P^0 = P_0 = \int \left( T^1_1 + T^2_2 + T^3_3 - T^0_0 \right) \sqrt{-g} \, dV. \tag{3.19}
\]

In the immediate vicinity of the mass, it is easy, to show that generalizing (3.19) leads to the 4-momentum vector that includes the right hand side of (3.12):

\[
P_a = \int \left[ (T^a_\text{matter}) + (T^a_\text{gravity}) \right] dS. \tag{3.20}
\]

Far from the source, we have obviously

\[
(P_a)_{\text{background field}} = \int \left[ (T^a_\text{background field}) \right] dS. \tag{3.21}
\]

where \( (T^a_\text{background field}) \) is a true tensor density, and the conservation law applied to \( P^a \) holds for all configurations, in accordance with (3.7) and (3.16).

4 Conclusions and outlook

In this short paper, we have sketched here a possible way out of the gravitational field pseudo-tensor.

From the beginning of General Relativity, the cosmological constant \( \Lambda \) has played an unsavory role. Einstein included this constant in his theory, because he wanted to have a cosmological model of the Universe which he wrongly thought static.

But to-day, a cosmological term seems to be badly needed to explain some astronomical observed clues, within the basic dynamical expanding model of Robertson-Walker [11], even though its occurrence was never clearly explained.

However, there is no reason \( a \) \( \text{a priori} \) to consider this cosmological term as constant everywhere.

In this respect, the background field hypothesis is rewarding in terms of several physical advantages:

- The ill-defined gravitational pseudo-tensor is now a true tensor, and it appears explicitly in the field equations with a massive source;
- The background persistent homogeneous energy field is then formally shown to be a consequence of the above derivation and it is actually regarded as the (sharply decreasing) continuation of any mass gravity field tensor;
- The inferred global energy-momentum tensor intrinsically satisfies the conservation law as well as the background field alone in the source free field equations, without introducing any other arbitrary ingredients or modification of the General Theory of Relativity.

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References


