Non-Linear Effects in Flow in Porous Duct

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In general, it is assumed in some non viscous flows that the flow velocity is constant at a cross-section. In this paper, we impose more realistic boundary conditions by, for example, introducing viscosity, and suction at walls, the net mass flow will change since the continuity equation must hold. The convective acceleration terms will be products of variables such that a non-linear behaviour will take place in the flow. The work will consist of deriving all the equations and parameters needed to described this kind of flow. An approximate analytic solution for the case of small Reynold number Re is discussed using perturbation techniques. Expression for the velocity components and pressure are obtained. The governing non-linear differential equation that cannot be solved analytically is solved numerically using Runge-Kutta Program and the graphs of axial and lateral velocity profiles are drawn.

1 Introduction

The problems of fluid flow through porous duct have arouse the interest of Engineers and Mathematicians, the problems have been studied for their possible applications in cases of membrane filtration, transpiration cooling, gaseous diffusions and drinking water treatment as well as biomedical engineering. Such flows are very sensitive to the Reynold number.

Berman was the first researcher who studied the problem of steady flow of an incompressible viscous fluid through a porous channel with rectangular cross section, when the Reynold number is low and the perturbation solution assuming normal wall velocity to be equal was obtained [1].

Sellars [2], extended the problem studied by Berman by using very high Reynold numbers.

Also wall suctions were recognize to stabilize the boundary layer and critical Reynold number for natural transition 46130 was obtained [3]. The stabilization effects of wall suction is due to the change of mean velocity profiles.

In the review of Joslin [4], it is also noticed that the uniform wall suction is not only a tool for laminar flow control but can also be used to damped out already existing turbulence.

The effects of Hall current on the steady Hartman flow subjected to a uniform suction and injection at the boundary plates has been studied [5].

Other reviews of flow in porous duct tend to focus only on one specific aspect of the subject at a time such as membrane filteration [8], the description of boundary conditions [6] and the existence of exact solutions [7].

In this paper, we consider the steady two-dimensional laminar flow of an incompressible viscous fluid between two parallel porous plates with equal suction and assume that the wall velocity is non uniform.

2 Formulation of the problem

The steady laminar flow of an incompressible viscous fluid between two parallel porous plates with an equal suction at walls and non uniform cross flow velocity is considered. The well known governing equations of the flow are:

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1}$$

Momentum equations (without body force)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{3}$$

Let us consider channel flow between uniformly parallel plates with equal suction. Assuming that we are far downstream of the entrance, the boundary conditions can be defined as

$$y = h, u = 0, v = v_w, \tag{4}$$

$$y = -h, u = 0, v = -v_w. \tag{5}$$

Let $\overline{u}(0)$ denote the average axial velocity at an initial section ($x = 0$). Then it is clear from a gross mass balance that $\overline{u}(x)$ will differ from $\overline{u}(0)$ by the amount $\frac{v_w}{h}x$. This observation led Berman(1953) to formulate the following relation for the stream in the channel [9].

$$\psi(x, y) = (h\overline{u}(0) - v_w x)\, f(y^*). \tag{6}$$

Where $y^* = \frac{y}{h}$, $\psi(x, y)$ is a stream function,$\overline{u}(0)$ is initial average axial velocity and f is dimensionless function to be determined. The velocity components follow immediately from the definition of ψ:

$$u(x,y^*) = \frac{\partial\psi}{\partial y} = \left(\overline{u}(0) - \frac{v_w x}{h}\right) f'(y^*) = \overline{u}(x) f'(y^*), \qquad (7)$$

$$v(x,y^*) = -\frac{\partial\psi}{\partial x} = v_w f(y^*) = v(y). \qquad (8)$$

The stream function must now be made to satisfy the momentum equations (2) and (3) for steady flow (2) and (3) will now become

$$u\frac{\partial u}{\partial x} + \frac{v}{h}\frac{\partial u}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2 u}{\partial y^{*2}}\right), \qquad (9)$$

$$u\frac{\partial v}{\partial x} + \frac{v}{h}\frac{\partial v}{\partial y^*} = -\frac{1}{\rho h}\frac{\partial p}{\partial y^*} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2 v}{\partial y^{*2}}\right). \qquad (10)$$

Using (7) and (8) in (9) and (10), the momentum equations reduces to,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \left(\overline{u}(0) - \frac{v_w x}{h}\right)\left(\frac{v_w}{h}\left(ff'' - f'^2\right) - \frac{\nu}{h^2} f'''\right), \qquad (11)$$

$$-\frac{1}{\rho h}\frac{\partial p}{\partial y^*} = \frac{v_w^2}{h} f f' - \frac{\nu v_w}{h^2} f''. \qquad (12)$$

Now differentiating (12) w.r.t x, we get

$$\frac{\partial^2 p}{\partial x \partial y^*} = \frac{\partial^2 p}{\partial x \partial y} = 0. \qquad (13)$$

Differentiating (11) w.r.t y^*, we get

$$\frac{\partial^2 p}{\partial x \partial y^*} = \left(\overline{u}(0) - \frac{v_w x}{h}\right)\frac{d}{dy^*}\left(\frac{v_w}{h}\left(ff'' - f'^2\right) - \frac{\nu}{h^2} f'''\right). \qquad (14)$$

From (13), (14) can be written as

$$\frac{d}{dy^*}\left(\frac{v_w}{h}\left(ff'' - f'^2\right) - \frac{\nu}{h^2} f'''\right) = 0, \qquad (15)$$

$$\frac{v_w}{h}\left(ff''' - f'f''\right) - \frac{\nu}{h^2} f'''' = 0.$$

Let the suction Reynold number be $Re = \frac{h v_w}{\nu}$ and substitute into above expression, we get

$$f'''' + Re\left(f'f'' - ff'''\right) = 0. \qquad (16)$$

(16) has no known analytic-closed form solution, but it can be integrated once i.e integrate (16) w.r.t y^*, we get

$$f''' + Re\left(f'^2 - ff''\right) = K = const. \qquad (17)$$

The boundary conditions on $f(y^*)$ 0f (4) and (5) can now be written as,

$$f(1) = 1, f(-1) = -1, f'(1) = 0, f'(-1) = 0. \qquad (18)$$

Hence, the solution of the equations of motion and continuity is given by non-linear fourth order differential equation (16) subject to the boundary condition (18).

3 Results

3.1 Approximate analytic solution (perturbation)

The non-linear ordinary differential equation (16) subject to condition (18) must in general be integrated numerically. However for special case when "Re" is small, approximate analytic results can be obtained by the use of a regular perturbation approach. Note that perturbation method has been used because the equations (16 and 18) are non-linear by using that technique, we get a linear approximated version of the true equations. The solution of $f(y^*)$ may be expanded in power of Re [10]

$$f(y^*) = \sum_{n=0}^{\infty} Re^n f_n(y^*) \qquad (19)$$

where $f_n(y^*)$ satisfies the symmetric boundary conditions

$$f_0(0) = f_0'(1) = f_0''(0) = 0, \quad f_0(1) = 1 \qquad (20)$$

and

$$f_n(0) = f_n'(1) = f_n''(0) = 0, \quad f_n(1) = 1. \qquad (21)$$

Here f_n are independent of Re. Substituting (19) in (16), we get

$$\left(f_0'''' + Ref_1'''' + Re^2 f_2''''\right) + Re\Big[\left(f_0' + Ref_1' + Re^2 f_2'\right)$$
$$\left(f_0'' + Ref_1'' + Re^2 f_2''\right) - \left(f_0 + Ref_1 + Re^2 f_2\right)$$
$$\left(f_0''' + Ref_1''' + Re^2 f_2'''\right)\Big] = 0.$$

Equating coefficients of Re, we get

$$f_o'''' = 0, \qquad (22)$$

$$f_1'''' + f_o' f_o'' - f_o f_o''' = 0, \qquad (23)$$

$$f_2'''' + f_o' f_1'' + f_1' f_o'' - f_o f_1''' - f_1 f_o''' = 0. \qquad (24)$$

The solution of (22) is of the form

$$f_o(y^*) = \frac{Ay^{*3}}{6} + \frac{By^{*2}}{2} + Cy^* + D,$$

where A,B,C and D are constants.

Applying the boundary condition (20) to the above equation, we get

$$f_o(y^*) = \frac{1}{2}\left(3y^* - y^{*3}\right). \qquad (25)$$

The solutions of Eq (23) and (24) subject to the boundary condition (21), are:

$$f_1(y^*) = -\frac{1}{280}\left(y^{*7} - 3y^{*3} - 2y^*\right), \qquad (26)$$

$$f_2(y^*) = \frac{1}{1293600} \times$$
$$\left(14y^{*11} - 385y^{*9} + 198y^{*7} + 876y^{*3} - 703y^*\right). \qquad (27)$$

Hence, the first order perturbation solution for $f(y^*)$ is

$$f'(y^*) = f_o(y^*) + Ref_1(y^*),$$

$$f^1(y^*) = \frac{1}{2}\left(3y^* - y^{*3}\right) - \frac{Re}{280}\left(y^{*^7} - 3y^{*3} - 2y^*\right). \quad (28)$$

The second order perturbation of solution for $f(y^*)$ is

$$f^2(y^*) = f_o(y^*) + Ref_1(y^*) + Re^2 f_2(y^*),$$

$$f^2(y^*) = \frac{1}{2}\left(3y^* - y^{*3}\right) - \frac{Re}{280}\left(y^{*7} - 3y^{*3} - 2y^*\right) + \frac{Re^2}{1293600}\left(14y^{*11} - 385y^{*9} + 198y^{*7} + 876y^{*3} - 703y^*\right). \quad (29)$$

Hence, the first order expression for the velocity components are:

$$u(x, y^*) = \left[\overline{u}(0) - \frac{v_w x}{h}\right] f'(y^*) = \left[\overline{u}(0) - \frac{v_w x}{h}\right] \frac{3}{2}\left(1 - y^{*2}\right)\left(1 - \frac{Re}{420}\left(2 - 7y^{*2} - 7y^{*4}\right)\right), \quad (30)$$

$$v(x, y^*) = v_w f(y^*) = v_w \left[\frac{1}{2}\left(3y^* - y^{*3}\right) - \frac{Re}{280}\left(y^{*7} - 3y^{*3} - 2y^*\right)\right]. \quad (31)$$

For pressure distribution, from Eq. (11) we get

$$\frac{h^2}{\rho\nu}\frac{\partial p}{\partial x} = \left[\overline{u}(0) - \frac{v_w x}{h}\right]\left[f'''(y^*) + Re\left(f'^2(y^*) - f(y^*)f''(y^*)\right)\right],$$

and since $f'''(y^*) + Re\left(f'^2(y^*) - f(y^*)\right) f''(y^*)) = K$, from (17), we have:

$$\frac{\partial p}{\partial x} = \frac{K\rho\nu}{h^2}\left[\overline{u}(0) - \frac{v_w x}{h}\right] = \frac{K\mu}{h^2}\left[\overline{u}(0) - \frac{v_w x}{h}\right]. \quad (32)$$

Now, from Eq. (12), we have

$$\frac{\partial p}{\partial y^*} = \frac{\mu v_w}{h} f''(y^*) - \rho v^2 f(y^*) f'(y^*). \quad (33)$$

Since $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y^*}dy^*$, then

$$dp = \frac{K\mu}{h^2}\left[\overline{u}(0) - \frac{v_w x}{h}\right]dx + \left[\frac{\mu v_w}{h} f''(y^*) - \rho v^2 f(y^*) f'(y^*)\right]dy^*. \quad (34)$$

Integrating (34), we get

$$p(x, y^*) = p(0,0) - \frac{\rho v^2}{2} f^2(y^*) + \frac{K\mu}{h^2}\left[\overline{u}(0)x - \frac{v_w x^2}{2h}\right] + \frac{\mu v_w}{h}\left[f'(y^*) - f'(0)\right]. \quad (35)$$

The pressure drop in the major flow direction is given by

$$p(x, 0) - p(x, y^*) = \frac{K\mu}{h^2}\left[\frac{v_w x^2}{2h} - \overline{u}(0)x\right]. \quad (36)$$

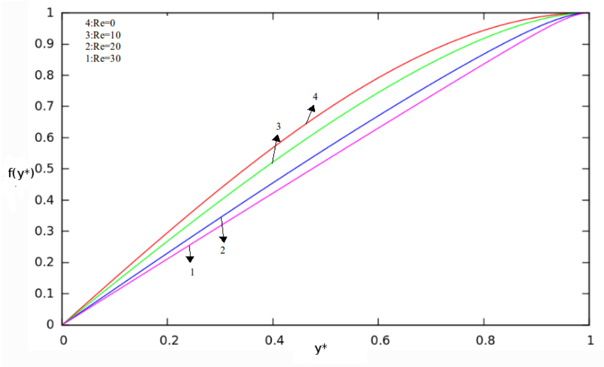

Fig. 1: Lateral velocity profiles for flow between parallel plates with equal suctions for different values of Re.

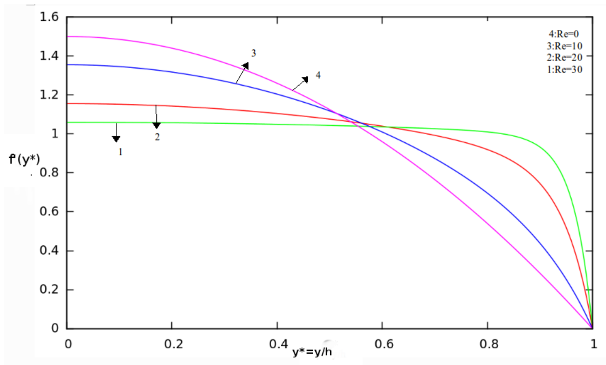

Fig. 2: Axial velocity profiles for flow between parallel plates with equal suctions for different values of Re.

3.2 Numerical solution

The approximate results of the previous section are not reliable when the Reynold number is not small. To obtain the detail information on the nature of the flow for different values of Reynold number (i.e. $Re = 0, 10, 20, 30$), a numerical solution to the governing equations is necessary. The Runge-Kutta program App.C is used to solve Eq. (17) numerically. One initial condition and constant (K) are unknown; i.e. starting at $y^* = 1$, then $f''(1)$ and K were guessed and the solution double-iterated until $f(-1) = -1$ and $f'(-1) = 0$. The most complete sets of profiles are shown in the figs. 1 and 2.

4 Discussion

The velocity profiles have been drawn for different values of Reynold number (i.e. $Re = 0, 10, 20, 30$). The shapes change smoothly with Reynold number and show no odd or unstable behaviour. Suction tends to draw the profiles toward the wall. From fig. (1), it is observed that for $Re > 0$ in the region $0 \leq y^* \leq 1$, $f(y^*)$ decreases with the increase of Reynold number Re. Also from fig. (2), it is observed that, for $Re > 0$, then $f'(y^*)$ decreases with an increase of the Reynold number in the range of $0 \le y^* \le 1$.

5 Conclusion

In this paper, a class of solutions of laminar flow through porous duct has been presented. Numerical approach is necessary for arbitrary values of Re. Also, when a cross flow velocity along the boundary is not uniform, a numerical technique is necessary to solve Eq. (2) and (3). Also, from the results obtained in this article, we can now conclude that, the non-linear effects of a flow of the porous duct is due to non uniform cross flow velocity and non vanishing terms of convective acceleration of momentum equations. The perturbation solution obtained for this problem reduces to the results of Berman [1].

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Nomenclature

A,B,C,D: Constants
K: Arbitrary Constant
f: Dimensionless function representing lateral velocity profile
h: Height of the channel
P: Pressure
x: Axial distance
y: Lateral distance
v_w: Lateral wall velocity
u(x,y): Axial velocity component
v(x,y): Lateral velocity component
$y^* = \frac{y}{h}$: Dimensionless lateral distance
$Re = \frac{v_w h}{\nu}$: Wall Reynold number

Greek Symbols

μ: Shear viscosity
ν: Kinematic viscosity
ρ: Fluid density
$\psi(x, y)$: Stream function.

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