Lorentzian Type Force on a Charge at Rest

Rudolf Zelsacher

Infineon Technologies AG, Siemensstrasse 2 A-9500 Villach, Austria. E-mail: Rudolf.zelsacher2@infineon.com

A remarkable achievement of theoretical physics is the explanation of magnetic effects, described by the Lorentz force, to be corollaries of charge invariance, Coulombs Law and the Lorentz transformation. The relativistic explanation of magnetism is based essentially on the calculation of Coulomb forces between moving charges in the laboratory reference system. We will show presently that the ideas used for the relativistic explanation of magnetism also lead to a force on a charge at rest by moving charges, which we dub "Lorentzian type force on a charge at rest".

1 Introduction

1.1 Miscellaneous

We will follow very closely the chain of thought taken by Edward Mills Purcell in [1]. We will use the Gaussian CGS units in order to underline the close relationship between electric field $\mathbf{E}(x, y, z, t)$ and magnetic field $\mathbf{B}(x, y, z, t)$. We will use as our reference frame $F[x, y, z, t]$, the idealized laboratory inertial frame, abbreviated to lab, to describe the location of particles and fields at time t. We will use other reference systems like $F'[x', y', z', t']$ with axes parallel with respect to F, with the origins of these systems coinciding at $t = t' = 0$ and with F' being in uniform relative motion with respect to F in either the positive or negative x direction.

Table 1: Definition of symbols

symbol	description
F	inertial frame/system
$\mathbf{F}$	also for force
$\mathbf{p}$	momentum
q	charge
$\mathbf{B}$	magnetic field
$\mathbf{E}$	electric field
a	surface
S	surface
(x, y, z)	space coordinates
t	time
c	speed of light in vacuum
v	velocity
I	current
l	length
β	$\frac{v}{c}$
γ	$\frac{1}{\sqrt{1-\beta^2}}$
m	rest mass
$\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\mathbf{r}}$	unit vector in the indicated direction

1.2 The charge and the mass of moving charged particles

The conclusion of the experimental findings is that charge is quantized and invariant in all stages of relative motion, and can be calculated by Gauss's Law [1]

$$q' = q\,. \tag{1}$$

Mass changes with velocity, charge does not. The fact that mass changes with velocity finds its mathematical formulation through the introduction of relativistic momentum [2]

$$\mathbf{p} = m\mathbf{v}\gamma \tag{2}$$

and relativistic energy $E = mc^2\gamma$. Eq. 2 is the starting point for the derivation of forces in inertial systems connected by the Lorentz transformation.

1.3 The electric fields E in F arising from a point charge q at rest in F' and moving with v in F

The electric field $\mathbf{E}$ in F of a charge moving uniformly in F, at a given instant of time, is generally directed radially outward from its instantaneous position and given by [1]

$$\mathbf{E}(\mathbf{R}, \vartheta) = \frac{q\left(1-\beta^2\right)}{R^2\left(1-\beta^2\sin^2\vartheta\right)^{\frac{3}{2}}}\,\hat{\mathbf{R}}\,. \tag{3}$$

R is the length of $\mathbf{R}$, the radius vector from the instantaneous position of the charge to the point of observation; ϑ is the angle between the direction of motion of the charge q $\mathbf{v}\Delta t$ and $\mathbf{R}$. Eq. 3, multiplied by Q, tells us the force on a charge Q at rest in F caused by a charge q moving in F (q is at rest in F').

1.4 The relativistic explanation of magnetism

In Fig. 1 we have sketched the model given in [1] to explain magnetic effects by relativistic arguments. The calculation of the force on q gives

$$F'_y = \frac{dp'_y}{dt'} = qE' = \frac{2q}{\gamma_0 r'}\left(\gamma'_+\lambda_+ + \gamma'_-\lambda_-\right) = \frac{\gamma 4q\lambda v v_0}{r'c^2}. \tag{4}$$

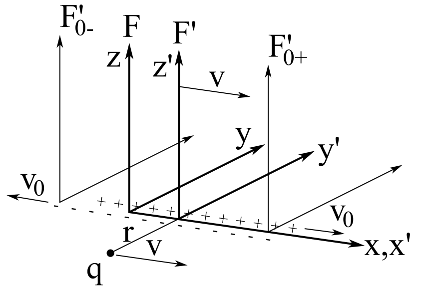


Fig. 1: We show a positive line charge distribution λ_+, stationary in reference frame F'_{0+}, moving in F in the positive x-direction with v_0, and a negative line charge distribution λ_- at rest in F'_{0-}, moving in F in the negative x-direction with v_0. A positive charge q, at rest in F', moves with v in F in the positive x-direction In F the electric fields sum up to **0** because by definition $\lambda_+ + \lambda_- = 0$. In F', the rest frame of charge q, there is an electric field $\mathbf{E'} \neq \mathbf{0}$ due to fields transformed from the rest frame of λ_+ and λ_- to F'. The resulting force in F' on q, $d\mathbf{p}'/dt'$, is then transformed to F, the *lab* frame, where we observe the charge q.

The resulting force $d\mathbf{p}'/dt'$ on charge q in F' is transferred to F, the *lab* system, where we do the experiments, giving $F_y = \frac{dp_y}{dt} = \frac{dp'_y}{\gamma dt'}$, and has the value [1]

$$\mathbf{F} = \frac{4q\lambda v v_0}{rc^2}\,\hat{\mathbf{y}} = \frac{2qvI}{rc^2}\,\hat{\mathbf{y}} = \frac{qv}{c}\frac{2I}{rc}\,\hat{\mathbf{y}} \qquad (5)$$

with $\lambda = |\lambda_+| = |\lambda_-|$.

As was discovered well before the advent of relativity, the overall effect of currents on a moving charge can also be described completely by introducing the magnetic field **B** in the *lab* frame F and equating the Lorentz force to $d\mathbf{p}/dt$. The magnetic field **B** is calculated with Biot-Savart's Law. The main purpose of the derivation, which results in Eq. 5, is to explain how nature works, and to demonstrate how the physical entity "magnetic field" can be revealed using more fundamental physical laws, specifically Coulomb's law and the laws of special relativity [1].

2 Lorentzian type force on a test charge Q at rest

We consider now two very narrow wires isolated along their length, but connected at the ends, each having length 2a and lying in lab coaxial to the x-axis of F from $x = -a$ to $x = a$. In addition the system has a source of electromotive force applied so that a current I is flowing through the wires: in one of the wires I flows in the positive x direction and in the other wire I flows in the negative x direction. We also have in mind two wires forming a thermocouple or two superconducting wires. On the z-axis of F fixed (at rest) at (0, 0, h) a test charge Q is located.

The system is sketched in Fig. 2. We will now calculate the force $\mathbf{F}_{Lt}$ on the stationary test charge Q fixed at (0, 0, h)

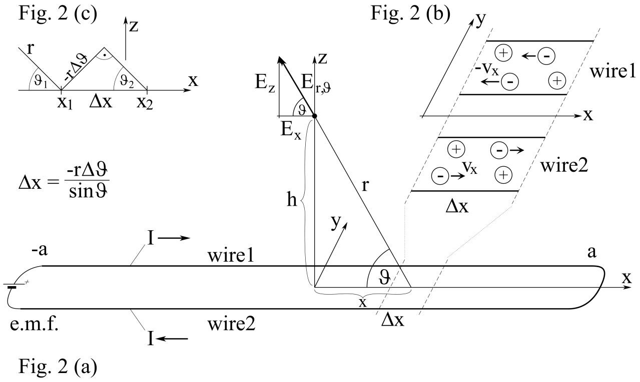
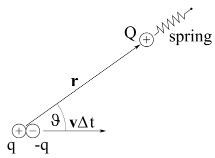


Fig. 2: (a) (b) (c): We show in Fig. 2(a) the two wires carrying the current I extended along the x axis of F from $x = -a$ to $x = a$ and the charge Q at rest in F at (0, 0, h). Additionally on the right-hand side a magnification of a small element Δx containing the two wires and labeled Fig. 2(b) can be seen. Fig. 2(b) shows some moving electrons and for each of these the nearest neighboring proton situated in the tiny element. We calculate the force on Q by precisely these pairs of charges. The effects of the other immobile electrons and protons of element Δx sum up to **0**. On the left-hand side another magnification of element Δx labeled Fig. 2(c) can be seen, showing some geometrical relations useful for integration.

due to the electrons of current I and their nearest stationary protons.

The two wires are electrically neutral before the current is switched on. Therefore after the current is switched on we have an equal number of N electrons and N protons in the system — the same number N, as with the current switched off. We look at the system at one instant of lab time t_0, after the current I is switched on and is stationary. We divide the wires into sections having lengths Δx_i. In each such element we consider the k_i electrons that make up the current I. For each of these k_i electrons e_{ij} with $j = 1, 2, \ldots k_i$, having velocity $\pm v_x$, which are defining the current I in Δx_i, we select the nearest neighboring stationary proton p_{ij} with $j = 1, 2, \ldots k_i$, with the restriction that the proton must lie in Δx_i. "Stationary" means that the charges retain their mean position over time. The effects on Q by the residual K_i stationary protons and K_i stationary electrons present in this element Δx_i sum up to **0**. The number of electrons and protons in the system is given by $N = \sum_i (K_i + k_i)$. For each charge of the mobile electron-stationary proton pairs present in Δx_i, we use the same $\mathbf{r}_i$ as the vector from each of the charges to Q. We use $\vartheta_i = \arcsin \frac{h}{r_i}$, the angle between the x-axis and $\mathbf{r}_i$, for each charge of the pairs of charges present in Δx_i. In Fig.3 we have sketched the situation for one pair of charges.

Referring to Fig. 2 we conclude that the line charge density λ and k_i, the number of current electrons moving with $|v_x|$ in Δx_i, and the line charge density λ and the k_i immobile



$$\mathbf{F}_Q = \frac{d\,(m\mathbf{v}_Q\gamma_Q)}{dt} =$$

$$-\frac{qQ}{r^2}\left(\frac{\left(1-\beta^2\right)}{\left(1-\beta^2\sin^2\vartheta\right)^{\frac{3}{2}}} - 1\right)\hat{\mathbf{r}} + \mathbf{F}_{spring} = \mathbf{0}$$

q $\mathbf{v} = \mathbf{0}$ at rest in laboratory

$-q$ $\mathbf{v} \neq \mathbf{0}$ moving in laboratory

Q $\mathbf{v}_Q = \mathbf{0}$ at rest in laboratory

$\beta = \frac{v^2}{c^2}$ $\gamma_Q = \left(1 - \frac{v_Q{}^2}{c^2}\right)^{-\frac{1}{2}}$

Fig. 3: We show a positive immobile charge q at rest in *lab* and a negative moving charge $-q$ moving in *lab* and the resulting electrical force on a positive charge Q at rest in *lab*. q and $-q$ are one of the pairs of charges that we select at time t_0 in Δx to calculate the effects of the current I on charge Q at rest in *lab*.

protons of Δx_i are both related by

$$k_i = \frac{|\lambda|\Delta x_i}{e} \tag{6}$$

with $e = 4.803^{-10}[esu]$. By doing so we replace the use of the relativistic length contraction by counting charges. We use the same distance $r_i(t_0)$ from Q to the k_i moving electrons and from Q to the k_i immobile protons. We now calculate the force on Q from exactly these charges, i.e. k_i electrons moving with $|v_x|$ and k_i immobile protons. In Figure 2(c) we sketched the model and some geometrical relations which are used below.

With

$$\Delta x = -\frac{r\Delta\vartheta}{\sin\vartheta} \tag{7}$$

and with

$$r = \frac{h}{\sin\vartheta} \tag{8}$$

we get

$$\Delta E_z = \frac{\lambda\left(1-\beta^2\right)\sin\vartheta\Delta\vartheta}{h\left(1-\beta^2\sin^2\vartheta\right)^{\frac{3}{2}}} - \frac{\lambda\sin\vartheta\Delta\vartheta}{h}\,. \tag{9}$$

Now we have to sum up over all elements Δx_i (or $\Delta\vartheta_i$). We do this by multiplying Eq. 9 by 4 and by integrating from $\vartheta = \frac{\pi}{2}$ to $\vartheta_{min} = \arctan\frac{h}{a}$. For the first term we substitute $u = \beta\cos\vartheta$ and use $\int \frac{du}{(a^2+u^2)^{\frac{3}{2}}} = \frac{u}{a^2(a^2+u^2)^{\frac{1}{2}}}$ and finally obtain

$$E_z = \frac{4\lambda\cos\vartheta_{min}}{h}\left(1 - \frac{1}{\left(1-\beta^2\sin^2\vartheta_{min}\right)^{\frac{1}{2}}}\right) \approx$$

$$-\frac{2Iv_x\cos\vartheta_{min}\sin^2\vartheta_{min}}{hc^2}.$$

The force on Q, — the “Lorentzian type force on a charge at rest” — is then

$$\mathbf{F}_{Lt} = -\frac{Q\,2Iv_x\cos\vartheta_{min}\sin^2\vartheta_{min}}{hc^2}\,\hat{\mathbf{z}}; \tag{10}$$

q.e.d.

The force described by Eq. 10 is of the same order (e.g. for $\vartheta_{min} = \frac{\pi}{3}$) of magnitude as magnetic forces, as can be seen by comparing it to Eq. 5 (repeated below), but it acts on a charge Q which has zero velocity. Find Eq. 5 written again below

$$\mathbf{F} = \frac{4q\lambda vv_0}{rc^2}\,\hat{\mathbf{y}} = \frac{2qvI}{rc^2}\,\hat{\mathbf{y}} = \frac{qv}{c}\frac{2I}{rc}\,\hat{\mathbf{y}} \qquad \text{(5 repeated)} \tag{11}$$

for easier comparison with Eq. 10.

Discussion

Whenever new concepts and ideas are introduced in physics, it is to be expected that they not only adequately explain the existing findings, but also enable new predictions that are falsifiable by experimental means. The Lorentz force leaves no room for a force on a charge at rest caused by moving charges, because the velocity of the charge at rest is, of course, zero. But the ideas and methods of special relativity, when used to explain magnetism, show that such a force — a force of moving charges which are part of a neutral piece of matter containing the same number of electrons and protons — exerted on a charge at rest, a certain distance away of the above mentioned piece of matter, is possible. We have shown this by reproducing the derivation of magnetism by relativistic arguments given in [1] step-by-step and applying it to our system of wires and charges. We could have calculated the fields and forces on Q in a reference system F' where Q is at rest and transformed the result to F or lab to formally and completely reproduce the derivation of magnetism using relativity, resulting in Eq. 5 as shown in [1] and section 1.4. But as Q is at rest in lab, and therefore at rest in reference frame F, we have calculated the effects on Q due to moving charges directly in F using Eq. 3. Of course we then no longer need to transfer the rate of change of momentum to F because it is directly given in the frame F in which Q is at rest. In addition we have replaced the line charge variations in different reference frames due to the Lorentz-Fitzgerald length contraction used in [1] by defining pairs of moving current electrons and their nearest neighbor immobile protons to calculate the effects on the charge Q. In other words we have replaced the use of the Lorentz-Fitzgerald contraction by counting charges, and counting is relativistically invariant. The basic idea for the calculation of $\mathbf{F}_{Lt}$ manifestations is the use of pairs of moving and immobile charges. If the Lorentzian type force on a charge at rest cannot be found by experiment, and we have no hint that it exists, at least the derivation leading to Eq. 3, written down in [1], should be subject to a revision.

Acknowledgements

I am grateful to Thomas Ostermann for typesetting the equations and to Andrew Wood for correcting the English.

Submitted on January 14, 2014 / Accepted on January 24, 2014

References

1. Purcell E. M. Electricity and Magnetism. McGraw-Hill Book Company, New York, 1964.
2. Kittel C. et al. Mechanics. 2nd Edition, McGraw-Hill Book Company, New York, 1973.