A Closed Universe Expanding Forever

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In a recent paper, the expression $a(t) = e^{\frac{H_0 T_0}{\beta}\left[\left(\frac{t}{T_0}\right)^{\beta}-1\right]}$ where $\beta = 0.5804$, was proposed for the expansion factor of our Universe. According to it, gravity dominates the expansion (*matter era*) until the age of $T_\star = 3.214$ Gyr and, after that, dark energy dominates (*dark energy era*) leading to an eternal expansion, no matter if the Universe is closed, flat or open. In this paper we consider only the closed version and show that there is an upper limit for the size of the radial comoving coordinate, beyond which nothing is observed by our fundamental observer, on Earth. Our observable Universe may be only a tiny portion of a much bigger Universe most of it unobservable to us. This leads to the idea that an endless number of other fundamental observers may live on equal number of Universes similar to ours. Either we talk about many Universes — Multiverse — or about an unique Universe, only part of it observable to us.

1 Introduction

The Cosmological Principle states that the Universe is spatially homogeneous and isotropic on a sufficiently large scale [1–7]. This is expressed by the Friedmann spacetime metric:

$$ds^2 = \mathfrak{R}^2(T_0)\, a^2(t) \left(d\psi^2 + f_k^2(\psi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) - c^2 dt^2, \tag{1}$$

where ψ, θ and ϕ are comoving space coordinates ($0 \leqslant \psi \leqslant \pi$, for closed Universe, $0 \leqslant \psi \leqslant \infty$, for open and flat Universe, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$), t is the proper time shown by any observer clock in the comoving system. $\mathfrak{R}(t)$ is the scale factor in units of distance; actually it is the modulus of the *radius of curvature* of the Universe. The proper time t may be identified as the cosmic time. The function $a(t)$ is the usual expansion factor

$$a(t) = \frac{\mathfrak{R}(t)}{\mathfrak{R}(T_0)}, \tag{2}$$

being T_0 the current age of the Universe. The term $f_k^2(\psi)$ assumes the following expressions:

$$f_k^2(\psi) \begin{Bmatrix} f_1^2(\psi) = \sin^2\psi \ \text{(closed Universe)} \\ f_0^2(\psi) = \psi^2 \ \text{(flat Universe)} \\ f_{-1}^2(\psi) = \sinh^2\psi \ \text{(open Universe)} \end{Bmatrix}. \tag{3}$$

In a previous paper [8], we have succeeded in obtaining an expression for the expansion factor

$$a(t) = e^{\frac{H_0 T_0}{\beta}\left(\left(\frac{t}{T_0}\right)^{\beta}-1\right)}, \tag{4}$$

where $\beta = 0.5804$ and H_0 is the so called Hubble constant, the value of the Hubble parameter $H(t)$ at $t = T_0$, the current age of the Universe. Expression (4) is supposed to be describing the expansion of the Universe from the beginning of the so called *matter era* ($t \approx 1.3 \times 10^{-5}$ Gyr, after the Big Bang). Right before that the Universe went through the so called *radiation era*. In reference [8] we consider only the role of the matter (baryonic and non-baryonic) and the dark energy.

In Figure 1 the behaviour of the expansion acceleration, $\ddot{a}(t)$, is reproduced [8]. Before $t = T_\star = 3.214$ Gyr, acceleration is negative, and after that, acceleration is positive. To perform the numerical calculations we have used the following values: $H_0 = 69.32$ km×s^{-1}×Mpc^{-1} $= 0.0709$ Gyr^{-1}, $T_0 = 13.772$ Gyr [9].

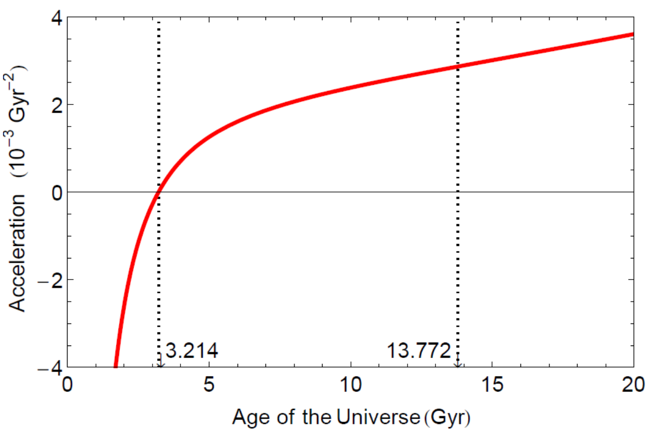

Fig. 1: $\ddot{a}(t) = a(t)\left(H_0\left(\frac{t}{T_0}\right)^{\beta} - (1-\beta)\frac{1}{t}\right)H_0\left(\frac{t}{T_0}\right)^{\beta-1}$.

2 The closed Universe

In reference [8], some properties such as Gaussian curvature $K(t)$, Ricci scalar curvature $R(t)$, matter and dark energy density parameters $(\Omega_m, \Omega_\lambda)$, matter and dark energy densities (ρ_m, ρ_λ), were calculated and plotted against the age of the Universe, for $k = 1, 0, -1$. It was found that the current curvature radius $\mathfrak{R}(T_0)$ has to be larger than 100 Gly. So, arbitrarily, we have chosen $\mathfrak{R}(T_0) = 102$ Gly. None of the results were sufficient to decide which value of k is more appropriate for the Universe. The bigger the radius of curvature, the less we can distinguish which should be the right k.

In this paper we explore only the $k = 1$ case (closed Universe). First, we feel it is appropriate to make the following consideration. At time $t \approx 3.8\times10^{-4}$ Gyr, after the Big Bang, the temperature of the universe fell to the point where nuclei could combine with electrons to create neutral atoms and photons no longer interacted with much frequency with matter. The universe became transparent, the cosmic microwave background radiation (CMB) erupted and the structure formation took place [10]. The occurrence of such CMB and the beginning of the matter era happen at different times, but, for our purpose here, we can assume that they occured approximately at the same time $t \approx 0$, since we will be dealing with very large numbers (billion of years). We have to set that our fundamental observer (Earth) occupies the $\psi = 0$ position in the comoving reference system. To reach him(her) at cosmic time T, the CMB photons spend time T since their emission at time $t \approx 0$, at a specific value of the comoving coordinate ψ. Let us call ψ_T this specific value of ψ. We are admitting that the emission of the CMB photons occured simultaneously ($t \approx 0$) for all possible values of ψ.

Having said that, we can write, for the trajectory followed by a CMB photon ($ds^2 = 0, d\phi = d\theta = 0$), the following:

$$-\frac{cdt}{\Re(t)} = d\psi, \tag{5}$$

$$-\int_0^T \frac{c}{\Re(t)}dt = \int_{\psi_T}^0 d\psi, \tag{6}$$

$$\psi_T = \frac{c}{\Re(T_0)}\int_0^T \frac{1}{a(t)}dt. \tag{7}$$

The events ($\psi = 0$, $t = T$) and ($\psi = \psi_T$, $t = 0$) are connected by a null geodesics. ψ gets bigger out along the radial direction and has the unit of angle.

The comoving coordinate which corresponds to the current "edge" (particle horizon) of our visible (observable) Universe is

$$\begin{aligned}\psi_{T_0} &= \frac{c}{\Re(T_0)}\int_0^{T_0}\frac{1}{a(t)}dt \\ &= \frac{c}{\Re(T_0)}\int_0^{T_0} e^{\frac{H_0T_0}{\beta}\left(1-\left(\frac{t}{T_0}\right)^\beta\right)}dt \\ &= 0.275 \text{ Radians} = 15.7 \text{ Degrees.}\end{aligned} \tag{8}$$

So CMB photons emitted at ψ_{T_0} and $t = 0$ arrive at $\psi = 0$ and $t = T_0$, the current age. Along their whole trajectory, other photons emitted, at later times, by astronomical objects that lie on the way, join the troop before reaching the fundamental observer. So he(she) while looking outwards deep into the sky, may see all the information "collected" along the trajectory of primordial CMB photons. Other photons emitted at the same time $t = 0$, at a comoving position $\psi > \psi_{T_0}$ will reach $\psi = 0$ at $t > T_0$, together with the other photons provenient from astronomical objects along the way. As the Universe gets older, its "edge" becomes more distant and its size gets bigger.

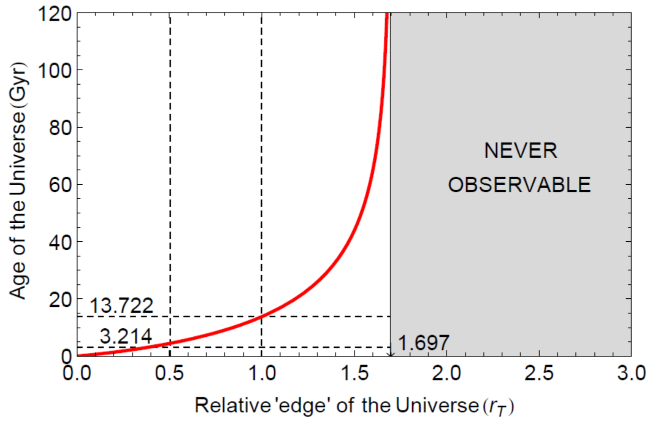

Fig. 2: $r_T = \int_0^T \frac{1}{a(t)}dt \,/\, \int_0^{T_0}\frac{1}{a(t)}dt$. The relative comoving coordinate r_T, from which CMB photons leave, at $t \approx 0$, and reach relative comoving coordinate $r = 0$ at age $t = T$ gives the relative position of the "edge" of the Universe ($r_{T\to\infty} \to 1.697$). (Axes were switched.)

The value of ψ depends on $\Re(T_0)$, the curvature radius. According to reference [8], it is important to recall that the current radius of curvature should be greater than 100 Gly and, in order to perform our numerical calculations, we choose $\Re(T_0) = 102$ Gly. The actual value for ψ_{T_0} should be, consequently, less than that above (equation (8)).

To get rid of such dependence on $\Re(T_0)$, we find convenient to work with the ratio r

$$r \equiv \frac{\psi}{\psi_{T_0}}, \tag{9}$$

which we shall call the relative comoving coordinate.

Obviously, at the age T, r_T is a relative measure of "edge" position with respect to the fundamental observer. For a plot of r_T see Figure 2.

3 Universe or Multiverse?

One question that should come out of the mind of the fundamental observer is: "Is there a maximum value for the relative comoving coordinate r?" What would be the value of r_∞?

By calculating r_∞, we get

$$r_\infty = \frac{\int_0^\infty \frac{1}{a(t)}dt}{\int_0^{T_0}\frac{1}{a(t)}dt} = \frac{47.558}{28.024} = 1.697. \tag{10}$$

To our fundamental observer (Earth), there is an upper limit for the relative comoving coordinate $r = r_\infty = 1.697$, beyond that no astronomical object can *ever* be seen. This should raise a very interesting point under consideration.

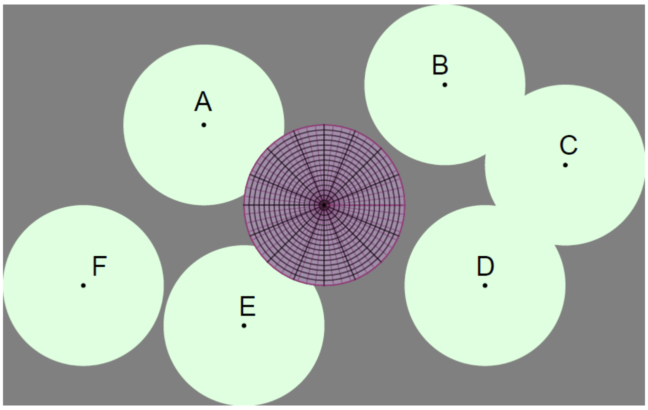

Fig. 3: This illustration tries to show schematically a hypersurface at time T with our Universe surrounded by other similar Universes, arbitrarily positioned, some of them overlapping.

Any other fundamental observer placed at a relative comoving coordinate $r > 2r_\infty$ ($\psi > 2\psi_\infty$), with respect to ours, will never be able to see what is meant to be our observable Universe. He (she) will be in the middle of another visible portion of a same whole Universe; He (she) will be thinking that he (she) lives in an observable Universe, just like ours. Everything we have been debating here should equally be applicable to such an "other" Universe.

The maximum possible value of ψ is π (equation (1)), then the maximum value of r should be at least 11.43. Just recall that $r = 1$ when $\psi = \psi_{T_0}$. This ψ_{T_0} was overevaluated as being 0.275 Radians = 15.7 Degrees, in equation (8) when considering the current radius of curvature as $\Re(T_0) = 102$ Gly. As found in reference [8] $\Re(T_0)$ should be bigger than that, not smaller. Consequently the real ψ_{T_0} should be smaller than 0.275 Radians = 15.7 Degrees. One direct consequence of this is that there is room for the ocurrence of a large number of isolated similar *observable Universes* just like ours.

We may say that the Big Bang gave birth to a large Universe, of which our current observable Universe is part, perhaps a tiny part. The rest is unobservable to us and an endless number of portions just the size of our visible Universe certainly exist, each one with their fundamental observer, very much probable discussing the same Physics as us.

Of course, we have to consider also the cases of overlapping Universes.

The important thing is that we are talking about one Universe, originated from one Big Bang, and that, contains many other Universes similar to ours. Would it be a *multiverse*? See Figure 3.

4 Proper distance, volume, recession speed and redshift

When referring to the relative coordinate r_T we are not properly saying it is a function of time. Actually r_T is the value of the relative comoving coordinate r from which the *CMB*

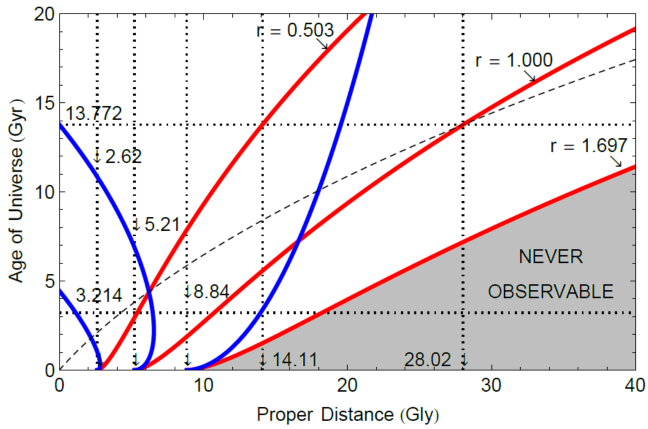

Fig. 4: Proper distances for $r = (0.503,\ 1.000,\ 1.697)$.
$d^{(r)}(T) = a(T) r d(T_0)$ (red curves),
$d(T) = a(T) r_T d(T_0)$ (dashed curve),
$d^{(r)}(T) - d(T) = a(T)(r - r_T) d(T_0)$ (blue curves).
Axes were switched for convenience.

photons leave, at $t \approx 0$, to reach our fundamental observer at cosmic time T. Because of the expansion of the Universe, the proper distance from our observer ($r = 0$) and a given point at $r > 0$, at the age t, is

$$d(t) = \Re(t) r \psi_{T_0} = a(t) c r \int_0^{T_0} \frac{1}{a(t')} dt'. \tag{11}$$

The proper distance from our observer ($r = 0$) to the farthest observable point ($r = r_T$), at the age T, is known as horizon distance:

$$d(T) = \Re(T) \int_0^{T} \frac{1}{\Re(t)} dt = a(T) c r_T \int_0^{T_0} \frac{1}{a(t)} dt. \tag{12}$$

Besides defining the "edge" of the observable Universe at age T, it is also a measure of its proper radius and does not depend on the radius of curvature. In Figure 4 it is the dashed curve. Its current value is

$$d(T_0) = c \int_0^{T_0} \frac{1}{a(t)} dt = 28.02 \text{ Gly}. \tag{13}$$

It will become $d(T \to \infty) \to \infty$. Although there is an upper value for r (or ψ), the proper radius of the Universe is not limited because of the continuous expansion (equation 1).

The proper distance from the observer to the position of arbitrarily fixed value of r is

$$d^{(r)}(T) = a(T) r d(T_0). \tag{14}$$

where $d(T_0)$ is given in equation (13). In Figure (4) we plot the age of the Universe as function of the proper distance, for three values of the relative comoving coordinate r (0.503, 1.000, 1.697) – red curves. Blue curves refer to null geodesics

$$d^{(r)}(T) - d(T) = a(T)(r - r_T) d(T_0) \tag{15}$$

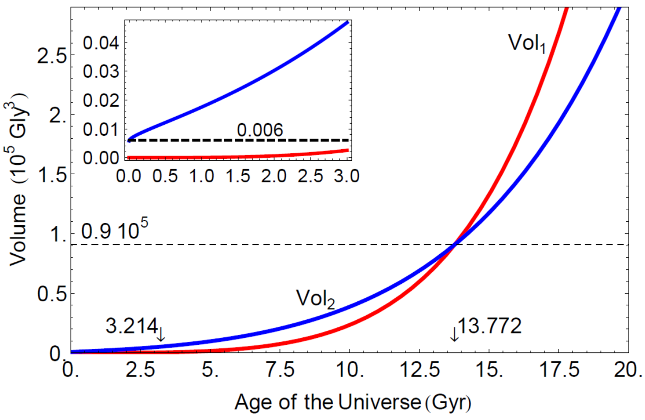

Fig. 5: Two evaluations of the volume of the Universe: $\mathrm{Vol}_1(T) = 2\pi\mathfrak{R}^3(T_0)a^3(T)(r_T\psi_{T_0} - \frac{1}{2}\sin 2r_T\psi_{T_0})$, $\mathrm{Vol}_2(T) = 2\pi\mathfrak{R}^3(T_0)a^3(T)(\psi_{T_0} - \frac{1}{2}\sin 2\psi_{T_0})$.

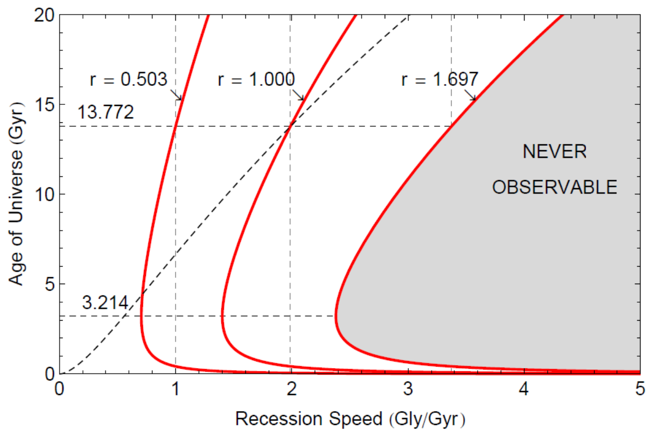

Fig. 6: $v(T) = a(T)H(T)rd(T_0)$. Recession speed is calculated for three values of the relative comoving coordinate r, as function of the age T of the Universe. For convenience the axes were switched.

for fixed values of $r \neq 0$. (The axes in Figure 4 are switched, for convenience.)

Consider the volume of our observable Universe. The general expression is

$$\begin{aligned}\mathrm{Vol}(t) &= \mathfrak{R}^3(T_0)a^3(t)\int_0^{\psi}\sin^2\psi d\psi\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi \\ &= 2\pi\mathfrak{R}^3(T_0)a^3(t)\left(\psi - \tfrac{1}{2}\sin 2\psi\right).\end{aligned} \tag{16}$$

Our fundamental observer may ask about two volumes:

First, the volume of the allways visible (observable) part since the beginning - such volume should be approximately zero for $t \approx 0$; Second, the volume of what became later the current visible part and that was not visible in its integrity in the past since $t \approx 0$. They are respectively,

$$\begin{aligned}\mathrm{Vol}_1(T) &= 2\pi\mathfrak{R}^3(T_0)a^3(T)(\psi - \tfrac{1}{2}\sin 2\psi) \\ &= 2\pi\mathfrak{R}^3(T_0)a^3(T)(r_T\psi_{T_0} - \tfrac{1}{2}\sin 2r_T\psi_{T_0}).\end{aligned} \tag{17}$$

$$\mathrm{Vol}_2(T) = 2\pi\mathfrak{R}^3(T_0)a^3(T)(\psi_{T_0} - \frac{1}{2}\sin 2\psi_{T_0}). \tag{18}$$

By evaluating equations (17 − 18) with $T = 0$, we get

$$\begin{aligned}\mathrm{Vol}_1(0) &= 0 \\ \mathrm{Vol}_2(0) &= 0.006 \times 10^5\mathrm{Gly}^3.\end{aligned} \tag{19}$$

These results are not surprising. To our observer, located at $r = 0$, at $t \approx 0$, the visible Universe is approximately zero, just because all the CMB photons are "born" at the same moment ($T = 0$); He (she) sees first the closest photons and then, in the sequence, the others as time goes on.

On the other hand,

$$\mathrm{Vol}_2(T_0) = \mathrm{Vol}_1(T_0) = 0.9 \times 10^5\mathrm{Gly}^3, \tag{20}$$

which is the volume of current observable Universe. See Figure 4. It is only about 150 times bigger than it was at $t = 0$.

Just one comment: If the reader wants to calculate the volume using the classical euclidean expression for the sphere ($(4\pi/3)\mathfrak{R}^3(T_0)a^3(t)\psi^3$), he (she) will get practically the same result. So here, as in reference [8], no distinction between $k = 0$ and $k = 1$.

The recession speed of a point of the Universe at a given relative comoving coordinate r, at cosmic time t, is

$$v(t) = a(t)H(t)rd(T_0), \tag{21}$$

where $\dot{a}(t)$ was replaced by

$$\dot{a}(t) = a(t)H(t), \tag{22}$$

and the Hubble parameter $H(t)$ is given by [8]

$$H(t) = H_0\left(\frac{t}{T_0}\right)^{\beta-1}. \tag{23}$$

The cosmological redshift is defined as

$$z = \frac{\Delta\lambda}{\lambda_e} = \frac{a(t_o)}{a(t_e)} - 1, \tag{24}$$

where λ_e and λ_o are, respectively, the photon wavelength at the source ($t = t_e$) and at the observer ($r = 0,\ t = t_o$). Due to expansion of the Universe, these two wavelengths are different. The redshift to be detected by the observer at $r = 0$, at current age should be

$$z = \frac{1}{a(t_e)} - 1 = e^{\frac{H_0T_0}{\beta}\left(1-\frac{t_e}{T_0}\right)^{\beta}} - 1. \tag{25}$$

The recession speed at coordinate r at time ($t = t_e$) is

$$v(t_e) = a(t_e)H(t_e)rd(T_0). \tag{26}$$

Fig. 7: $v(z) = \left(1 - \frac{\beta}{H_0 T_0}\text{Log}(1+z)\right)^{\beta - \frac{1}{\beta}} \frac{H_0 r}{1+z} d(T_0)$. Recession speeds calculated as function of the cosmological redshift and plotted with switched axes, for convenience.

From equation (25) we obtain

$$t_e = T_0 \left(1 - \frac{\beta}{H_0 T_0}\text{Log}(1+z)\right)^{\frac{1}{\beta}}, \tag{27}$$

which inserted into equation (24) gives

$$v(z) = \left(1 - \frac{\beta}{H_0 T_0}\text{Log}(1+z)\right)^{\beta - \frac{1}{\beta}} \frac{H_0 r}{1+z} d(T_0). \tag{28}$$

Because of the transition from negative to positive expansion acceleration phenomenon, we have, in many situations, two equal recession speeds separated in time leading to two different redshifts. See Figure 7.

5 Conclusion

The expansion factor $a(t) = e^{\frac{H_0 T_0}{\beta}\left(\left(\frac{t}{T_0}\right)^{\beta} - 1\right)}$, where $\beta = 0.5804$ [8], is applied to our Universe, here treated as being closed ($k = 1$). We investigate properties of comoving coordinates, proper distances, volume and redshift under the mentioned expansion factor. Some very interesting conclusions were drawn. One of them is that the radial relative comoving coordinate r, measured from the fundamental observer, $r = 0$ (on Earth), to the "edge" (horizon) of our observable Universe has an upper limit. We found that $r \to 1.697$ when $T \to \infty$. Therefore all astronomical objects which lie beyond such limit would never be observed by our fundamental observer ($r = 0$). On the other hand any other fundamental observer that might exist at $r > 2\times 1.697$ would be in the middle of another Universe, just like ours; he(she) would never be able to observe our Universe. Perhaps he(she) might be thinking that his(her) Universe is the only one to exist. An endless number of other fundamental observers and an equal number of Universes similar to ours may clearly exist. Situations in which overlapping Universes should exist too. See Figure 3.

The fact is that the Big Bang originated a big Universe. A small portion of that is what we call our observable Universe. The rest is unobservable to our fundamental observer. Equal portions of the rest may be called also Universe by their fundamental observers if they exist. So we may speak about many Universes - a Multiverse - or about only one Universe, a small part of it is observable to our fundamental observer.

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