On the Applicability of Bell's Inequality

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We investigate the applicability of Bell's inequality based on the assumptions used in its derivation. We find that it applies to a specific class of hidden variable theories referred to as Bell theories, but not necessarily to other hidden variable dynamic theories. We consider examples of quantum dynamical processes that cannot be represented by the initial representation defined in Bell's derivation. We highlight two hidden assumptions identified by Jaynes [11] that limit the applicability of Bell's inequality, as derived, to Bell hidden variable theories and that show that there are no superluminal physical influences, only logical inferences.

1 Introduction

Bell's inequality [1–3] sets constraints for the existence of local hidden variable theories in quantum mechanics. Bohr, of the Copenhagen probabilistic school, and Einstein, of the objective reality school, who both contributed to the foundation of quantum mechanics, did not agree on its interpretation – their views and correspondence on the topic are well documented in many books [4–7].

In 1935, Einstein, Podolsky and Rosen published a paper [8] that aimed to show that quantum mechanics was not a complete description of physical reality. Bohr provided a response to the challenge [9], but the EPR paper remained an argument for hidden variables in quantum mechanics. In 1964, Bell [1] published an inequality that imposed constraints for local hidden variable theories to be valid in quantum mechanics. The experiments performed by Aspect *et al* [10] with entangled photons confirmed that Bell's inequality was violated within experimental errors, taken to mean that local hidden variable theories are not valid in quantum mechanics. Only non-local hidden variable theories are possible, based on these results.

In this paper, we investigate the applicability of Bell's inequality, based on the assumptions used in its derivation.

2 Bell's inequality

Bell's derivation [1] considers a pair of spin one-half particles of spin σ_1 and σ_2 respectively, formed in the singlet state, and moving freely in opposite directions. Then $\sigma_1 \cdot \mathbf{a}$ is the measurement of the component of σ_1 along some vector $\mathbf{a}$, and similarly for $\sigma_2 \cdot \mathbf{b}$ along some vector $\mathbf{b}$. Bell then considers the possibility of a more complete description using hidden variable parameters λ.

He writes down the following equation for the expectation value of the product of the two components $\sigma_1 \cdot \mathbf{a}$ and $\sigma_2 \cdot \mathbf{b}$ with parameters λ:

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{a}, \lambda)\, B(\mathbf{b}, \lambda) \qquad (1)$$

where

$$A(\mathbf{a}, \lambda) = \pm 1 \text{ and } B(\mathbf{b}, \lambda) = \pm 1 \qquad (2)$$

and $\rho(\lambda)$ is the probability distribution of parameter λ. This should equal the quantum mechanical expectation value

$$<\sigma_1 \cdot \mathbf{a}\; \sigma_2 \cdot \mathbf{b}> = -\mathbf{a} \cdot \mathbf{b}\,. \qquad (3)$$

Bell says that it does not matter whether λ is "a single variable or a set, or even a set of functions, and whether the variables are discrete or continuous" [1]. He uses a single continuous parameter described by a probability distribution. In a later paragraph, he states that (1) represents all kinds of possibilities, such as any number of hidden variables, two sets of hidden variables dependent on A and B, or even as initial values of the variables λ at a given time if one wants to assign "dynamical significance and laws of motion" [1] to it. However, it is doubtful that the probability distribution $\rho(\lambda)$ can be used to represent all possible theories of hidden variables.

Indeed, the basic limitation of (1) with its use of a probability distribution $\rho(\lambda)$ is that it imposes a quantum mechanical calculation representation on the analysis. Other quantum level dynamic theories, which we will refer to as hidden variable dynamic theories, could obey totally different dynamic principles, in which case, (1) would not be applicable. Equation (1) is only applicable to a specific class of hidden variable theories that can be represented by that equation, which Jaynes [11] refers to as Bell theories. In the following sections, we consider examples of quantum dynamical processes that cannot be represented by (1) or by the probability distribution $\rho(\lambda)$ used in (1).

3 Measurement limitations and inherent limitations

It is important to note that Bohr's responses to Einstein's *gedanken* experiments were based on measurements arguments, which acted as a barrier to any further analysis beyond that consideration. As pointed out by Jaynes [12], Einstein and Bohr "were both right in the essentials, but just thinking on different levels. Einstein's thinking [was] always on the ontological level traditional in physics; trying to describe the realities of Nature. Bohr's thinking [was] always on the epistemological level, describing not reality but only our information about reality".

As discussed in [13], the Heisenberg Uncertainty Principle arises because x and p form a Fourier transform pair of variables at the quantum level due to the momentum p of a quantum particle being proportional to the de Broglie wave number k of the particle. It is a characteristic of quantum mechanics that conjugate variables are Fourier transform pairs of variables.

It is thus important to differentiate between the measurement limitations that arise from the properties of Fourier transform pairs, and any inherent limitations that may or may not exist at the quantum level for those same variables, independently of the measurement process. Conjugate variable measurement limitations affect how we perceive quantum level events as those can only be perceived by instrumented measurements at that level. However, as shown in [13], conjugate variable measurement limitations affect *only* our perception of the quantum environment, and are *not* inherent limitations of the quantum level.

The Nyquist-Shannon Sampling Theorem of Fourier transform theory allows access to the range of values of variables below the Heisenberg Uncertainty Principle limit under sampling measurement conditions, as demonstrated by the Brillouin zones formulation of solid state physics [13] [14, see p. 21] [15, see p. 100]. Physically this result can be understood from the sampling measurement operation building up the momentum information during the sampling process, up to the Nyquist limit. This shows that there are local hidden variables at the quantum level, independently of the measurement process. The dynamical process in this case is masked by the properties of the Fourier transform.

4 Wave-particle duality in *STCED*

The Elastodynamics of the Spacetime Continuum (*STCED*) [16] has similarities to Bohmian mechanics in that the solutions of the *STCED* wave equations are similar to Louis de Broglie's "double solution" [17, 18]. Bohmian mechanics also known as de Broglie-Bohm theory [19–21] is a theory of quantum physics developed by David Bohm in 1952 [22], based on Louis de Broglie's original work on the *pilot wave*, that provides a causal interpretation of quantum mechanics. It is empirically equivalent to orthodox quantum mechanics, but is free of the conceptual difficulties and the metaphysical aspects that plague the interpretation of quantum theory.

Interestingly, Bell was aware of and a proponent of Bohmian mechanics when he derived his inequality [23]:

> "Bohm showed explicitly how parameters could indeed be introduced, into nonrelativistic wave mechanics, with the help of which the indeterministic description could be transformed into a deterministic one. More importantly, in my opinion, the subjectivity of the orthodox version, the necessary reference to the 'observer,' could be eliminated... I will try to present the essential idea... so compactly, so lucidly, that even some of those who know they will dislike it may go on reading, rather than set the matter aside for another day."

In Bohmian mechanics, a system of particles is described by a combination of the wavefunction from Schrodinger's equation and a guiding equation that specifies the location of the particles. "Thus, in Bohmian mechanics the configuration of a system of particles evolves via a deterministic motion choreographed by the wave function" [21] such as in the two-slit experiment. We will see a similar behavior in the *STCED* wave equations below. Bohmian mechanics is equivalent to a non-local hidden variables theory.

In the Elastodynamics of the Spacetime Continuum, as discussed in [24], energy propagates in the spacetime continuum by longitudinal (*dilatation*) and transverse (*distortion*) wave displacements. This provides a natural explanation for wave-particle duality, with the transverse mode corresponding to the wave aspects of the deformations and the longitudinal mode corresponding to the particle aspects of the deformations.

The displacement u^ν of a deformation from its undeformed state can be decomposed into a longitudinal component $u_\parallel^\nu$ and a transverse component $u_\perp^\nu$. The volume dilatation ε is given by the relation $\varepsilon = u_\parallel{}^\mu{}_{;\mu}$ [16]. The wave equation for $u_\parallel^\nu$ describes the propagation of longitudinal displacements, while the wave equation for $u_\perp^\nu$ describes the propagation of transverse displacements in the spacetime continuum. The u^ν displacement wave equations can be expressed as a longitudinal wave equation for the dilatation ε and a transverse wave equation for the rotation tensor $\omega^{\mu\nu}$ [16].

Particles propagate in the spacetime continuum as longitudinal wave displacements. Mass is proportional to the volume dilatation ε of the longitudinal mode of the deformation [16, see (32)]. This longitudinal mode displacement satisfies a wave equation for ε, different from the transverse mode displacement wave equation for $\omega^{\mu\nu}$. This longitudinal dilatation wave equation for ε is given by [16, see (204)]

$$\nabla^2 \varepsilon = -\frac{\bar{k}_0}{2\bar{\mu}_0 + \bar{\lambda}_0}\, u_\perp^\nu \varepsilon_{;\nu} \tag{4}$$

where $\bar{\mu}_0$ and $\bar{\lambda}_0$ are the Lamé constants and $\bar{k}_0$ the elastic volume force constant of the spacetime continuum. It is important to note that the inhomogeneous term on the R.H.S. includes a dot product coupling between the transverse displacement and the volume dilatation for the solution of the longitudinal dilatation wave equation for ε.

The transverse distortion wave equation for $\omega^{\mu\nu}$ [16, see (210)]

$$\nabla^2 \omega^{\mu\nu} + \frac{\bar{k}_0}{\bar{\mu}_0}\, \varepsilon(x^\mu)\, \omega^{\mu\nu} = \frac{1}{2}\frac{\bar{k}_0}{\bar{\mu}_0}\left(\varepsilon^{;\mu} u_\perp^\nu - \varepsilon^{;\nu} u_\perp^\mu\right) \tag{5}$$

also includes a R.H.S. coupling, in this case a cross product, between the transverse displacement and the volume dilatation for the solution of the transverse distortion wave equation for $\omega^{\mu\nu}$. The transverse distortion wave $\omega^{\mu\nu}$ corresponds to a multi-component wavefunction Ψ.

A deformation propagating in the spacetime continuum consists of a combination of longitudinal and transverse waves. The coupling between $\varepsilon^{;\mu}$ and $u^{\nu}_{\perp}$ on the R.H.S. of both wave equations explains the behavior of electrons in the double slit interference experiment. It shows that even though the transverse wave is the source of the interference pattern in double slit experiments, the longitudinal dilatation wave, which behaves as a particle, follows the interference pattern dictated by the transverse distortion wave as observed experimentally. The longitudinal dilatation wave behaves as a particle and goes through one of the slits, even as it follows the interference pattern dictated by the transverse distortion wave, as observed experimentally [25, see in particular Figure 4] and as seen in the coupling between $\varepsilon^{;\mu}$ and $u^{\nu}_{\perp}$ in (4) and (5) above. This behavior is the same as that in Bohmian mechanics seen above. These results are in agreement with the results of the Jánossy-Naray, Clauser, and Dagenais and Mandel experiments on the self-interference of photons and the neutron interferometry experiments performed by Bonse and Rauch [26, see pp. 73-81].

As mentioned previously, the solutions of the *STCED* wave equations are similar to Louis de Broglie's "double solution". The longitudinal wave is similar to the de Broglie "singularity-wave function" [17]. In *STCED* however, the particle is not a singularity of the wave, but is instead characterized by its mass which arises from the volume dilatation ε propagating as part of the longitudinal wave. There is no need for the collapse of the wavefunction Ψ, as the particle resides in the longitudinal wave, not the transverse one. A measurement of a particle's position is a measurement of the longitudinal wave, not the transverse wave.

In addition, $|\Psi|^2$ represents the physical energy density of the transverse (*distortion*) wave. It corresponds to the transverse field energy of the deformation. It is not the same as the particle, which corresponds to the longitudinal (*dilatation*) wave displacement and is localized within the deformation via the massive volume dilatation. However, $|\Psi|^2$ can be normalized with the system energy and converted into a probability density, thus allowing the use of the existing probabilistic formulation of quantum theory.

The dynamical process, although it has some similarities to Bohmian mechanics, is also different from it as it is centered on longitudinal (particle) and transverse (wavefunction) wave equations derived from the properties of the spacetime continuum of general relativity. It is thus deterministic and causal as is general relativity.

5 Physical influence versus logical inference

We have considered two examples of quantum dynamical processes where the starting equation (1) and the probability distribution $\rho(\lambda)$ used in (1) do not apply to the situation. We now examine in greater details the probabilistic formulation of Bell's inequality derivation of section 2 to better understand its limitations.

Physicist E. T. Jaynes was one of the proponents of the usage of probability theory as an extension of deductive logic. His textbook "Probability Theory: The Logic of Science" [27] published posthumously is an invaluable resource for scientists looking to understand the scientific use of probability theory as opposed to the conventional mathematical measure theory. As he states in [11],

> "Many circumstances seem mysterious or paradoxical to one who thinks that probabilities are real physical properties existing in Nature. But when we adopt the "Bayesian Inference" viewpoint of Harold Jeffreys [28,29], paradoxes often become simple platitudes and we have a more powerful tool for useful calculations."

Jaynes clarifies this approach to probability theory and contrasts it to frequencies as follows [11]:

> "In our system, a probability is a theoretical construct, on the epistemological level, which we assign in order to represent a state of knowledge, or that we calculate from other probabilities according to the rules of probability theory. A frequency is a property of the real world, on the ontological level, that we measure or estimate."

The probability distributions used for inference do not describe a property of the world, only a certain state of information about the world, which provides us with the means to use prior information for analysis as powerfully demonstrated in numerous applications in [11, 12, 27].

The Einstein–Podolsky–Rosen (EPR) paradox and Bell inequality in quantum theory is one of the examples examined by Jaynes in [11]. In quantum mechanics, the belief that probabilities are real physical properties leads to quandaries such as the EPR paradox which lead some to conclude that there is no real world and that physical influences travel faster than the speed of light, or worse ("a spooky kind of action at a distance" as Einstein called it). As Jaynes points out, it is important to note that the EPR article did not question the existence of the correlations, which were expected, but rather the need for a physical causation instead of what he calls "instantaneous psychokinesis", based on experimenter decisions, to control distant events.

Jaynes' analysis of the derivation of Bell's inequality uses the following notation for conditional probabilities which corresponds to Bell's notation as follows:

$$P(AB\,|\,ab) = P(\mathbf{a},\mathbf{b}) \tag{6}$$

$$P(A\,|\,a\lambda) = A(\mathbf{a},\lambda)\,, \tag{7}$$

such that Bell's equation (1) above becomes

$$P(AB\,|\,ab) = \int d\lambda\,\rho(\lambda)\,P(A\,|\,a\lambda)\,P(B\,|\,b\lambda)\,. \tag{8}$$

However, as Jaynes notes, the fundamentally correct relation for $P(AB\,|\,ab)$ according to probability theory should be

$$P(AB\,|\,ab) = \int d\lambda\, P(AB\,|\,ab\lambda)\, P(\lambda\,|\,ab)\,. \qquad (9)$$

Assuming that knowledge of the experimenters' choices gives no information about λ, then one can write

$$P(\lambda\,|\,ab) = \rho(\lambda)\,. \qquad (10)$$

The fundamentally correct factorization of the other probability factor of (9), $P(AB\,|\,ab\lambda)$, is given by [11]

$$P(AB\,|\,ab\lambda) = P(A\,|\,ab\lambda)\, P(B\,|\,Aab\lambda)\,. \qquad (11)$$

However, as Jaynes notes, one could argue as Bell did that EPR demands that A should not influence events at B for space-like intervals. This requirement then leads to the factorization used by Bell to represent the EPR problem

$$P(AB\,|\,ab\lambda) = P(A\,|\,a\lambda)\, P(B\,|\,b\lambda)\,. \qquad (12)$$

Nonetheless, the factorization (12) disagrees with the formalism of quantum mechanics in that the result of the measurement at A must be known before the correlation affects the measurement at B, *i.e.* $P(B\,|\,Aab)$. Hence it is not surprising that Bell's inequality is not satisfied in systems that obey quantum mechanics.

Two additional hidden assumptions are identified by Jaynes in Bell's derivation, in addition to those mentioned above:

1. Bell assumes that a conditional probability $P(X\,|\,Y)$ represents a physical causal influence of Y on X. However, consistency requires that conditional probabilities express logical inferences not physical influences.
2. The class of Bell hidden variable theories mentioned in section 2 does not include all local hidden variable theories. As mentioned in that section, hidden variable theories don't need to satisfy the form of (1) (or alternatively (8)), to reproduce quantum mechanical results, as evidenced in Bohmian mechanics.

Bell's inequality thus applies to the class of hidden variable theories that satisfy his relation (1), *i.e.* Bell hidden variable theories, but not necessarily to other hidden variable dynamic theories.

The superluminal communication implication stems from the first hidden assumption above which shows that what is thought to travel faster than the speed of light is actually a logical inference, not a physical causal influence. As summarized by Jaynes [11],

> "The measurement at A at time t does not change the real physical situation at B; but it changes our state of knowledge about that situation, and therefore it changes the predictions we are able to make about B at some time t'. Since this is a matter of logic rather than physical causation, there is no action at a distance and no difficulty with relativity."

There is simply no superluminal communication, as required by special relativity. Assuming otherwise would be similar to Pauli assuming that the established law of conservation of energy mysteriously fails in weak interactions instead of successfully postulating a new particle (the neutrino).

6 Discussion and conclusion

In this paper, we have investigated the applicability of Bell's inequality, based on the assumptions used in its derivation. We have considered two examples of hidden variable dynamic theories that do not satisfy Bell's initial equation (1) used to derive his inequality, and consequently for which Bell's inequality is not applicable: one based on the Nyquist-Shannon Sampling Theorem of Fourier transform theory and the other based on the wave-particle solutions of the *STCED* wave equations which are similar to Louis de Broglie's "double solution". We highlight two hidden assumptions identified by Jaynes [11] that limit the applicability of Bell's inequality, as derived, to Bell hidden variable theories and that show that there are no superluminal physical influences, only logical inferences.

We close with a quote from Jaynes [27, see p. 328] that captures well the difficulty we are facing:

> "What is done in quantum theory today... when no cause is apparent one simply postulates that no cause exists – ergo, the laws of physics are indeterministic and can be expressed only in probability form."

Thus we encounter paradoxes such as seemingly superluminal physical influences that contradict special relativity, and "spooky action at a distance" is considered as an explanation rather than working to understand the physical root cause of the problem. This paper shows that, in this case, the root cause is due to improper assumptions, specifically the first hidden assumption identified by Jaynes highlighted in section 5 above, that is assuming that a conditional probability represents a physical influence instead of the physically correct logical inference. In summary,

> "He who confuses reality with his knowledge of reality generates needless artificial mysteries." [11]

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