Dark Matter, the Correction to Newton's Law in a Disk

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The dark matter problem in the context of spiral galaxies refers to the discrepancy between the galactic mass estimated from luminosity measurements of galaxies with a given mass-to-luminosity ratio and the galactic mass measured from the rotational speed of stars using the Newton's law. Newton's law fails when applied to a star in a spiral galaxy. The problem stems from the fact that Newton's law is applicable to masses represented as points by their barycenter. As spiral galaxies have shapes similar to a disk, we shall correct Newton's law accordingly. We found that the Newton's force exerted by the interior mass of a disk on an adjacent mass shall be multiplied by the coefficient η_{disk} estimated to be 7.44 ± 0.83 at a 99% confidence level. The corrective coefficient for the gravitational force exerted by a homogeneous sphere at it's surface is 1.00 ± 0.01 at a 99% confidence level, meaning that Newton's law is not modified for a spherical geometry. This result was proven a long time ago by Newton in the shell theorem.

1 Introduction

Dark matter is an hypothetical type of matter, which refers to the missing mass of galaxies, obtained from the difference between the mass measured from the rotational speed of stars using the Newton's law and the visual mass. The visual mass is estimated based on luminosity measurements of galaxies with a given mass-to-luminosity ratio.

The problem of galaxy rotational curves was discovered by Vera Rubin in the 1970s [1–3], with the assistance of the instrument maker Kent Ford. In Figure 1, we show the rotational velocity curve of stars versus the expected rotational velocity curve from visible mass as a function of the radius of a typical spiral galaxy. According to [4], the estimated dark matter to visible matter ratio in the universe is about 5.5.

It has been hypothesized that dark matter is made of invisible particles which do not interact with electromagnetic radiations. The hunt for the dark matter particle has already begun. The Xenon dark matter experiment [5] is taking place in a former gold mine nearly a mile underground in South Dakota. The idea is to find hypothetical dark matter particles underneath the earth to avoid particule interference from the surface.

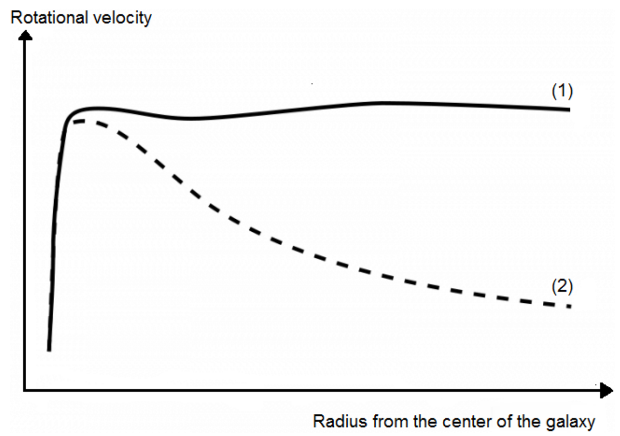

Fig. 1: The problem of galaxy rotational curves, where (1) is the actual rotational velocity curve of stars; and (2) the expected rotational velocity curve from the visible disk.

Other experiments seek dark matter in space. In 2011, NASA lauched the AMS (Alpha Magnetic Spectrometer) experiment, a particle detector mounted on the ISS (International Space Station) aimed at measuring antimatter in cosmic rays and search for evidence of dark matter. In December 2015, the Chinese Academy of Sciences lauched the DAMPE (Dark Matter Particle Explorer), a satellite hosting a poweful space telescope for cosmic ray detection and investigating particles in space and hypothetical dark matter.

An investigation of the amount of planetary-mass dark matter detected via gravitational microlensing concluded that these objects only represent a small portion of the total dark matter halo [6]. The study of the distribution of dark matter in galaxies led to the development of two models of the dark matter halo. These models are known as the dark matter halo profile of Navarro, Frenk and White [7], and the Burkert dark matter halo profile [8, 9].

Dark matter is a hot topic in particle physics, and has led to the development of various theories. According to [10], the favoured candidates for dark matter are axions, supersymmetric particles, and to some extent massive neutrinos. The Majorana fermion has also been proposed as a candidate for dark matter [11, 12]. Other candidates for dark matter would be dark pions, a set of pseudo-Goldstone bosons [13]. Many alternatives have been proposed including modified Newtonian gravity. Mordehai Milgrom proposed the MOND theory, according to which Newton's law is modified for large distances [14, 15]. Moffat proposed a modified gravity theory based on the action principle using field theory [16, 17]. James Feng and Charles Gallo proposed to model galaxy ro-

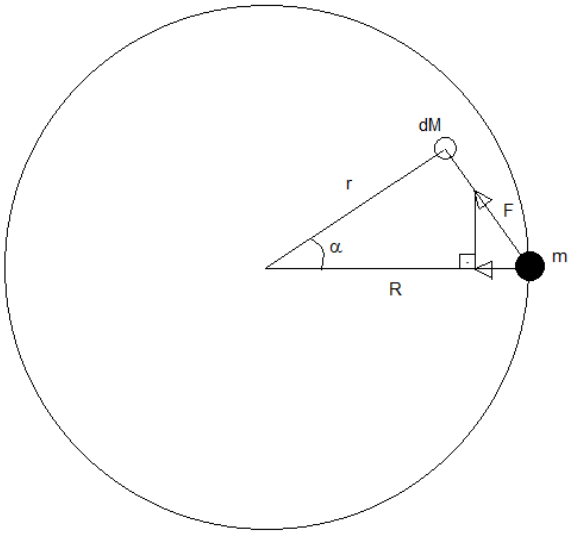

Fig. 2: Force exerted by an infinitesimal mass dM of the disk on a mass m located at the edge of the disk using polar coordinates. The radius of the disk is R. Let the mass dM be at a distance r from the center of the disk. Let α be the angle between the two axis passing by the center of the disk in the direction of the two masses dM and m.

tational curves by applying Newtonian dynamics to a rotating thin disk [18, 19]. Their approach is similar to the route we undertake in the current work, although the latter was done independently.

According to Pavel Kroupa, the dark matter crisis is a major problem for cosmology [20]. In addition, he states that the hypothesis that exotic dark matter exists must be rejected [21]. In the present study we find that dark matter is mainly a problem of geometry because Newton's law is applicable to masses which can be approximated by a point in space. Below, we compute the corrective coefficient to Newton's law in a disk and in a sphere.

2 Calculation of the gravitational force in a disk

The Newton's law states that the gravitational force between two bodies is expressed as follows:

$$F_{\text{Newton}} = \frac{G\,M\,m}{R^2}\,, \tag{1}$$

where G is the gravitational constant, M and m the respective masses of the two bodies in interaction, and R the distance between the barycenters of the two masses.

The shape of spiral galaxies allows us to use the gravitational force computed for a disk. Let us assume a homogeneous disk of surface density ρ_s, and radius R. A mass m is located at the edge of this disk at a distance R from the center of the disk.

In Figure 2, we represent the force exerted by an infinitesimal mass dM of the disk on the mass m using polar coordinates. Because of the symmetry of the disk with respect to the axis passing between its center and the mass m, we need to compute the projection of the force exerted by the infinitesimal mass dM on this axis. For this purpose we apply basic trigonometric rules (see figure 3). For convenience, we consider the polar coordinates (r, α) to describe the position of dM, where r is the radial distance, and α the angle between the mass dM and an arbitrary direction as viewed from the center of the disk.

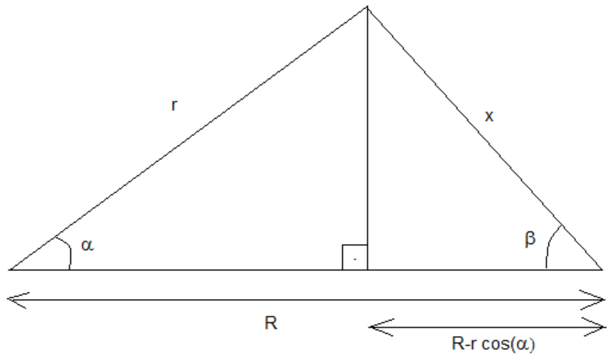

Fig. 3: Triangle to compute the projection of the force exerted by the infinitesimal mass dM on mass m on the axis passing by the center of the disk to the mass m

Let us say x is the distance between the mass dM and m. From trigonometry we calculate x as follows:

$$x^2 = r^2 \sin^2\alpha + (R - r\cos\alpha)^2\,. \tag{2}$$

Hence, we get:

$$x^2 = r^2 + R^2 - 2Rr\cos\alpha\,. \tag{3}$$

Let β be the angle between the center of the disk and the mass dM as viewed from the mass m. The angle β is calculated as follows:

$$\cos\beta = \frac{R - r\cos\alpha}{x}\,. \tag{4}$$

By Newton's law, the infinitesimal force exerted by dM on m projected on the axis passing through the center of the disk and the mass m is as follows:

$$dF = \frac{G\,m\,dM}{x^2}\cos\beta\,. \tag{5}$$

Combining (4) and (5), we get:

$$dF = \frac{G\,m\,dM}{x^3}\,(R - r\cos\alpha)\,. \tag{6}$$

Combining (3) and (6), we get:

$$dF = \frac{G\,m\,dM\,(R - r\cos\alpha)}{(r^2 + R^2 - 2Rr\cos\alpha)^{\frac{3}{2}}}\,. \tag{7}$$

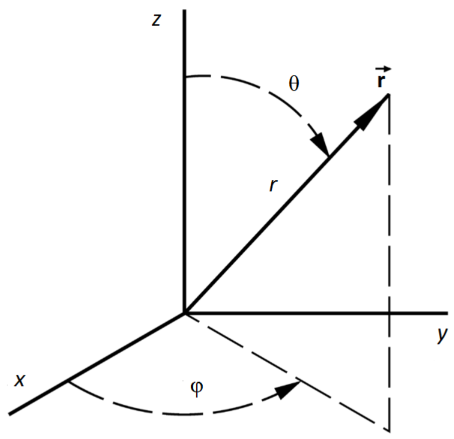

Fig. 4: Spherical coordinate system, where r is the radial distance, θ the polar angle, and φ the azimutal angle.

Because we are using polar coordinates, the surface element dA is as follows:

$$dA = r\,dr\,d\alpha\,. \tag{8}$$

To obtain the infinitesimal mass dM, we multiply the infinitesimal surface dA by the surface density ρ_s; hence, we get:

$$dM = \rho_s\,r\,dr\,d\alpha\,. \tag{9}$$

Therefore, the infinitesimal force dF is as follows:

$$dF = \frac{\rho_s\,G\,m\left(Rr - r^2\cos\alpha\right)}{(r^2 + R^2 - 2Rr\cos\alpha)^{\frac{3}{2}}}\,dr\,d\alpha\,. \tag{10}$$

Because the total mass of the disk is $M = \rho_s\,\pi\,R^2$, we get:

$$dF = \frac{G\,M\,m}{\pi\,R^2}\frac{\left(Rr - r^2\cos\alpha\right)}{(r^2 + R^2 - 2Rr\cos\alpha)^{\frac{3}{2}}}\,dr\,d\alpha\,. \tag{11}$$

The total force F exerted by the disk on the mass m is obtained by the following integral:

$$F = \frac{G\,M\,m}{\pi\,R^2}\int_{r=0}^{R}\int_{\alpha=0}^{2\pi}\frac{\left(Rr - r^2\cos\alpha\right)}{(r^2 + R^2 - 2Rr\cos\alpha)^{\frac{3}{2}}}\,dr\,d\alpha\,. \tag{12}$$

We rearrange the terms in the integral to obtain:

$$F = \frac{G\,M\,m}{\pi\,R^2}\times$$
$$\times\int_{r=0}^{R}\int_{\alpha=0}^{2\pi}\frac{R^2\left(\frac{r}{R} - \left(\frac{r}{R}\right)^2\cos\alpha\right)}{R^3\left(\left(\frac{r}{R}\right)^2 + 1 - 2\left(\frac{r}{R}\right)\cos\alpha\right)^{\frac{3}{2}}}\,dr\,d\alpha\,. \tag{13}$$

Hence:

$$F = \frac{G\,M\,m}{\pi\,R^3}\times$$
$$\times\int_{r=0}^{R}\int_{\alpha=0}^{2\pi}\frac{\left(\frac{r}{R} - \left(\frac{r}{R}\right)^2\cos\alpha\right)}{\left(\left(\frac{r}{R}\right)^2 + 1 - 2\left(\frac{r}{R}\right)\cos\alpha\right)^{\frac{3}{2}}}\,dr\,d\alpha\,. \tag{14}$$

We apply the change of variable $u = \frac{r}{R}$, hence $dr = R\,du$. Therefore, we get:

$$F = \frac{G\,M\,m}{\pi\,R^2}\int_{u=0}^{1}\int_{\alpha=0}^{2\pi}\frac{\left(u - u^2\cos\alpha\right)}{(u^2 + 1 - 2u\cos\alpha)^{\frac{3}{2}}}\,du\,d\alpha\,. \tag{15}$$

From (15), we see that in a disk, Newton's force $F_{\text{Newton}} = \frac{G\,M\,m}{R^2}$ needs to be multiplied by the following coefficient:

$$\eta_{disk} = \frac{1}{\pi}\int_{u=0}^{1}\int_{\alpha=0}^{2\pi}\frac{\left(u - u^2\cos\alpha\right)}{(u^2 + 1 - 2u\cos\alpha)^{\frac{3}{2}}}\,du\,d\alpha\,. \tag{16}$$

3 Calculation of the gravitational force in a sphere

Let us consider a homogeneous sphere of radius R and average mass density ρ. We consider an infinitesimal mass dM of the sphere represented by its spherical coordinates (r, θ, φ), where r is the radial distance, θ the polar angle, and φ the azimuthal angle (see Figure 4). Let the volume of the sphere be defined by the following boundaries: $r \in [0, R]$, $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi]$. We assume that a mass m is located at the surface of this sphere on the x-axis.

In Cartesian coordinates we have $x = r\sin\theta\,cos\varphi$, $y = r\sin\theta\,sin\varphi$ and $z = r\cos\theta$. Hence, the distance x between the mass dM and m is as follows:

$$x = \sqrt{(R - r\sin\theta\,\cos\varphi)^2 + r^2\sin^2\theta\,\sin^2\varphi + r^2\cos^2\theta}\,. \tag{17}$$

Let β be the angle as viewed from the mass m between the direction of the center of the sphere and the mass dM. Hence, we get:

$$\cos\beta = \frac{R - r\sin\theta\cos\varphi}{x}\,. \tag{18}$$

The volume element in spherical coordinates is as follows:

$$dV = r^2\sin\theta\,d\theta\,d\varphi\,dr\,. \tag{19}$$

Therefore, the infinitesimal force exerted by dM on m projected in the axis passing through m and the center of the sphere is as follows:

$$dF = \frac{G\,m\,\rho\,r^2 \sin\theta\,\cos\beta}{x^2}\,d\theta\,d\varphi\,dr = \frac{G\,m\,\rho\,r^2 \sin\theta\,(R - r\sin\theta\cos\varphi)}{x^3}\,d\theta\,d\varphi\,dr\,. \tag{20}$$

Let $M = \rho\frac{4}{3}\pi R^3$ be the total mass of the sphere, hence:

$$F = G\,m\,M\frac{3}{4\pi R^3}\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\frac{r^2\sin\theta\;(R - r\sin\theta\,\cos\varphi)}{\left(R^2 + r^2\sin^2\theta\,\cos^2\varphi - 2Rr\sin\theta\,\cos\varphi + r^2\sin^2\theta\,\sin^2\varphi + r^2\cos^2\theta\right)^{\frac{3}{2}}}\,d\theta\,d\varphi\,dr\,. \tag{21}$$

We rearrange the terms in the integral to obtain a function of ratios of r/R, and apply the substition $u = \frac{r}{R}$; hence, we get:

$$F = \frac{G\,m\,M}{R^2}\frac{3}{4\pi}\int_{u=0}^{1}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\frac{u^2\sin\theta\;(1 - u\sin\theta\cos\varphi)}{\left(1 + u^2\sin^2\theta\,\cos^2\varphi - 2u\sin\theta\,\cos\varphi + u^2\sin^2\theta\,\sin^2\varphi + u^2\cos^2\theta\right)^{\frac{3}{2}}}\,d\theta\,d\varphi\,dr\,. \tag{22}$$

Therefore, the corrective coefficient to Newton's law in a sphere is as follows:

$$\eta_{sphere} = \frac{3}{4\pi}\int_{u=0}^{1}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\frac{u^2\sin\theta\;(1 - u\sin\theta\,\cos\varphi)}{\left(1 + u^2\sin^2\theta\,\cos^2\varphi - 2u\sin\theta\,\cos\varphi + u^2\sin^2\theta\,\sin^2\varphi + u^2\cos^2\theta\right)^{\frac{3}{2}}}\,d\theta\,d\varphi\,dr\,. \tag{23}$$

4 Numerical evaluation of the gravitational corrective coefficients

Because the integrals in (16) and (23) do not have a known closed-form solution, we need to evaluate them numerically. Monte Carlo simulation is an appropriate method for computing multidimentional integrals. Using Monte Carlo simulation we can compute both an estimate of the integral and its standard deviation.

4.1 Numerical evaluation of the double integral over the disk

Let us consider the integration of a function $f(r,\alpha)$ over a disk of radius R in polar coordinates, where r is the radius and α an angle from a reference direction. The integral to evaluate is expressed as follows:

$$\int_0^{2\pi}\int_0^{R} f(r,\alpha)\,r\,dr\,d\alpha\,. \tag{24}$$

We shall apply the following change of variables:

$$\alpha = 2\pi u_1\,, \tag{25}$$

and

$$r = R\sqrt{u_2}\,, \tag{26}$$

where u_1 and u_2 are two independent random variables of uniform distribution over [0, 1]. This change of variables gives a uniform distribution on the disk of radius R.

Let N be the number of times we generate the random set (u_1,u_2). Hence, the integral of $f(r,\alpha)$ over the disk converges towards the following estimate for N large:

$$I = \pi R^2\frac{\sum_1^N f_i}{N}\,, \tag{27}$$

where f_i is the function $f(r,\alpha)$ evaluated for each draw of the random set (u_1,u_2) with the change of variables (25) and (26).

Because the variance of a random variable X is given by $Var(X) = E[X^2] - (E[X])^2$ and the variance of the sample mean is $Var(\bar{X}) = \frac{Var(X)}{N}$, the variance of the estimate is computed as follows:

$$\mathrm{Var}(I) = \frac{\pi^2R^4\frac{\sum_1^N f_i^2}{N} - \left(\pi R^2\frac{\sum_1^N f_i}{N}\right)^2}{N}\,. \tag{28}$$

The standard deviation of the estimate of η_{disk} is equal to the square root of the variance of the estimate of the double integral on the disk divided by π. To evaluate the integral in (16), we used the Mersenne Twister pseudo-random number generator [22] with N= 1.2×10^{10}. We obtained $\eta_{disk} = 7.44$ with standard deviation of 0.320.

4.2 Numerical evaluation of the triple integral over the sphere

As for the disk, let us use Monte Carlo simulation to evaluate the triple integral of $f(r,\theta,\varphi)$ over the sphere of radius R in the spherical coordinate system. The integral to evaluate is expressed as follows:

$$\int_0^{R}\int_0^{\pi}\int_0^{2\pi} f(r,\theta,\varphi)\,r^2\,\sin\theta\,d\varphi\,d\theta dr\,. \tag{29}$$

For this pupose we generate a set of three independent random variables (u_1,u_2,u_3), each with a uniform distribution over the interval [0, 1]. We apply the following change of variables, which gives a uniform distribution over the sphere:

$$\theta = 2\arcsin\left(\sqrt{u_1}\right)\,, \tag{30}$$

and

$$\varphi = 2\pi u_2\,, \tag{31}$$

and

$$r = R\,u_3^{\frac{1}{3}}\,. \tag{32}$$

Let N be the number of time we generate the random set (u_1, u_2, u_3). Hence, the triple integral over the sphere converges towards the following estimate for N large:

$$I = \frac{4\pi R^3}{3} \frac{\sum_1^N f_i}{N}\,, \tag{33}$$

where f_i is the function $f(r, \theta, \varphi)$ evaluated for each draw of the random set (u_1, u_2, u_3) using the change of variables (30), (31) and (32).

The variance of the estimate is computed as follows:

$$\mathrm{Var}(I) = \frac{\left(\frac{4\pi R^3}{3}\right)^2 \frac{\sum_1^N f_i^2}{N} - \frac{4\pi R^3}{3}\left(\frac{\sum_1^N f_i}{N}\right)^2}{N}\,. \tag{34}$$

The standard deviation of the estimate of η_{sphere} is equal to the square root of the variance of the estimate of the triple integral on the sphere multiplied by $\frac{3}{4\pi}$. To evaluate the integral in (23), we used the Mersenne Twister pseudo-random number generator with N= $1{\times}10^8$. We obtained $\eta_{sphere} = 1.00$ with standard deviation of $3.85{\times}10^{-3}$.

5 Interpretation

In the present study, we have solved the dark matter puzzle in the context of spiral galaxies by considering the geometry of massive bodies. Dark matter is a hypothetical mass introduced to fill the discrepancy between galaxy mass as measured from the rotational speed of stars and visible mass. Isaac Newton proved the shell theorem [23], which applies to objects of spherical geometry. The shell theorem states that:

1. A spherical body affects external objects gravitationally as though all of its mass were concentrated in a point at its barycenter.
2. For a spherical body, no net gravitational force is exerted by the external shell on any object inside the sphere, regardless of the position.

Because spiral galaxies have shapes which can be approximated by a disk, the distribution of matter will directly affect the perceived gravitational force for a mass rotating on such a disk, and the shell theorem does not apply. By considering an interior mass distributed in space according to an idealised homogeneous disk, we found that Newton's law is corrected by a multiplicative coefficient. This coefficient is estimated to be about 7.44 based on our calculations above of the dark matter to visible mass ratio of 5.5. This coefficient can be interpreted as if the mass of the disk was excentered towards the object perceiving it. In our calculations, we only considered the interior mass of the disk for radii below the position of the object. For an object located on the disk, the outer mass of the disk for radii above of the position of the object may also exert a gravitational force of opposite direction on the object, mitigating the gravitational force exerted by the interior of the disk. This effect which was not quantified should create the asymptotic behavior for galaxy rotational curves when moving far away from the galaxy's central bulge.

Furthermore, for a spiral galaxy, the mass density may increase as we move closer to the center of the disk, causing a departure from the idealised homogeneous disk. In addition, the closer we move towards the central supermassive black hole, which is spherical, the more the interior mass tends towards a sphere and the gravitational corrective coefficient converges towards unity. The shift in the gravitational corrective coefficient at different radii on the galactic disk ought to explain the observed shape of galaxy rotational curves.

Let us illustrate the impact of the gravitational coefficient we found on the mass of the Milky Way. The centripetal force of a star in orbit is expressed as $F_c = \frac{mv^2}{R}$, where m is the mass of the star, v the tangential velocity and R the radius to the center of the galaxy. Hence, the interior mass of the galaxy for a given star is expressed as $M = \frac{Rv^2}{\eta G}$, where $v = wR$ with w the angular velocity, η the gravitational coefficient, and G the gravitational constant. The apparent mass of the Milky Way was estimated to be around $6.82{\times}10^{11}\ M_\odot$ [24]. Let us approximate the Milky Way by a homogeneous disk; therefore, the gravitational coefficient at the periphery of the disk is about $\eta = 7.44$. This leads to an intrinsic mass of the Milky Way of $9.17{\times}10^{10} M_\odot$.

6 Conclusion

To address the discrepancy between galaxy mass estimated from the rotational velocity of stars and visual mass estimated from luminosity measurements, the existence of dark matter was hypothesized. A number of approaches have taken to hunt for both the dark matter particle and modified gravity. For instance, Milgrom proposed that Newton's law should be modified for large distances. Dark matter remains an unresolved problem challenging cosmology and particle physics.

In the present study, we propose a geometrical approach as Newton's law applies to masses that can be approximated by a point in space corresponding to their barycenter. As spiral galaxies have shapes close to a disk, we derived the corrective coefficient to Newton's law in an idealised disk of homogeneous mass distribution. We found that the Newton's law in a homogeneous disk shall be multiplied by the coefficient η_{disk} estimated to be 7.44 ± 0.83 at a 99% confidence level, which fills the dark matter gap in galaxy haloes. We conclude that dark matter in spiral galaxies is a problem of geometry, and that Newton's law needs to be corrected to account for the geometry of the mass. For a spherical geometry, we found that the corrective gravitational coefficient η_{sphere} is

1.00 ± 0.01 at a 99% confidence level.

This means that the Newton's law is not modified for spherical geometry, which was proven a long time ago by Newton.

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