Seeliger's Gravitational Paradox and the Infinite Universe

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Seeliger's paradox is often regarded as an argument against Newtonian potentials in an infinite universe. In this paper the argument is analyzed with the help of Riemann's series theorem. This theorem reveals that the paradox is a known consequence of the rearrangement of conditionally convergent series or integrals, and so it demonstrates that the same situation would arise with almost any other type of gravitational force law. Therefore Seeliger's argument is not a valid proof against Newton's inverse square law or even an infinite universe.

1 Introduction

In 1895 the German astronomer Hugo Seeliger published an article [1] in which he revealed an apparent flaw in Newton's law of gravitation, which may lead to "unsolvable contradictions". His reasoning can be presented as follows.

Let's suppose a boundless universe with a (near) homogeneous distribution of matter. For simplicity, let's assume this to be a continuous mass distribution, which extends uniformly to infinity in all directions. To calculate the gravitational force exerted by this infinite universe on a test particle with gravitational mass m located at a point P, we consider all the masses in the universe as arranged in thin concentric spheres centered in P. Since the Newtonian attraction of a sphere on any point located inside of it is zero, we find that the sum of all the concentric spheres extending to an infinite distance will be zero. This is what might be expected from symmetry.

Next, let's calculate the force again, but this time using a coordinate system centered at another point Q, located at an arbitrary distance d from m. In order to calculate the force, we divide the universe into two parts. The first one is the sphere of radius d centered on Q and passing through P. The mass of this sphere is $M = \frac{4}{3}\rho\pi d^3$, where ρ is its density, which attracts the material point m with a force given by $F = -\frac{GMm}{d^2} = \frac{4}{3}\rho\pi d^3$ pointing from P to Q. The second part is the remainder of the universe. This remainder is composed of a series of external shells also centered on Q containing the internal test particle m. As we have seen above, this second part exerts no force on m. Therefore the force exerted by the universe calculated in this way is proportional to the distance d and directed towards Q.

This means that depending on which point Q we choose, we obtain a different value for the force acting on m. The conclusion that Seeliger extracts from this puzzling result is that either the universe cannot be infinite, or that Newton's law of attraction must be modified. Taking the latter choice, he proposed to add an absorption factor $e^{-\lambda r}$ to the force of gravity

$$F_{Seeliger} = -G\,\frac{mm'}{r^2}\,e^{-\lambda r} \tag{1}$$

where λ is an arbitrary parameter, sufficiently small to make this force compatible with the existing observational data.

When (1) is used, it can be demonstrated [2] that the gravitational force exerted on a particle m at the surface of a spherical volume V_1 uniformly filled with matter is equal and opposite to the gravitational force exerted on the particle by all the infinite concentric uniform spherical shells outside the first spherical volume V_1, so that the net force acting on the particle is zero. Seeliger thus believed to have found a solution of the paradox.

The purpose of this paper is to generalize the formulation of the problem and to show that Seeliger's conclusion does not hold.

2 Newton's inverse square law and its relation to the paradox

Before getting at the origin of the paradox, let's look at different ways to formulate it.

First we note that Seeliger uses the fact, unique to the inverse square law, that the attraction of a sphere to any mass inside of it is zero. To demonstrate that this is not an essential feature, we will present the paradox from a different perspective.

Let's calculate the gravitational field of an infinite plane. Let ρ denote the mass density per unit area of this infinite plane and consider a test particle of mass m located at a distance h from the plane, as shown in the following figure.

In Newtonian terms, the incremental force dF on this par-

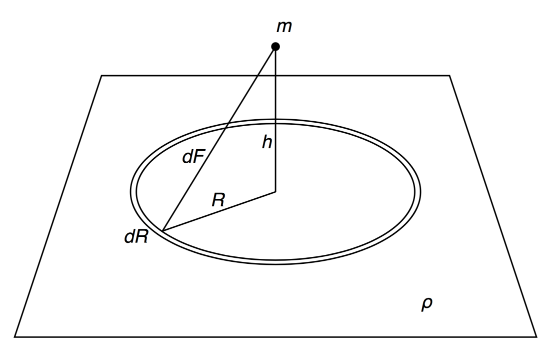

Fig. 1: Attraction of an infinite plane on a mass m.

ticle contributed by an annular ring of radius R and incremental width dR is just the projection onto the perpendicular of the forces exerted by each element of the plane around the circumference of the annular region. Thus we have:

$$dF = \frac{Gm\rho}{h^2 + r^2}\left[\pi(R + dR)^2 - \pi R^2\right]\left(\frac{h}{\sqrt{h^2 + R^2}}\right).$$

Expanding this expression and ignoring second order differential terms, we get

$$dF = 2\pi Gm\rho h\,\frac{R}{(h^2 + R^2)^{\frac{3}{2}}}\,dR\,.$$

Integrating from $R = 0$ to ∞, we find that the total force experienced by the particle is

$$F = 2\pi Gm\rho h\int_0^\infty \frac{R}{(h^2 + R^2)^{\frac{3}{2}}}\,dR = 2\pi Gm\rho\,.$$

Thus the force exerted on the particle is independent of the distance h from the plane. Adding more planes to form a slab of thickness a, we get that the force would be in this case:

$$F = 2\pi Gm\rho a.$$

Grouping infinite parallel slabs of the same thickness a and adding the contribution of each of them, we get the force of the universe acting on particle m

$$F = \sum_{-\infty}^{\infty} 2\pi Gm\rho a\,. \tag{2}$$

It can be shown that this infinite sum will yield a different result depending on how it is calculated. As a first way of determining the value of (2), let's pair each slab with its corresponding symmetrical one around the plane of origin. If we consider this plane as the plane xy, then we take a parallel slab of coordinate z_0 and pair it with the slab of coordinate $-z_0$. Since the force of each slab in the pair is equal and opposite, their sum vanishes. The total force (2) will thus be zero. Analytically, we can write this as

$$\begin{aligned} F = (2\pi G\rho ma - 2\pi G\rho ma) + \\ + (2\pi G\rho ma - 2\pi G\rho ma) + ... = 0\,. \end{aligned} \tag{3}$$

Next, let's calculate (2) again but this time starting one slab further from m. The total force on m will be the sum of the force due to this separate slab, which contains m on one of its surfaces, plus all the remaining slabs in the universe, on both sides of the first slab, thus

$$F = 2\pi G\rho ma_0 + \sum_{n=-\infty}^{0} 2\pi G\rho ma_n - \sum_{n=1}^{\infty} 2\pi G\rho ma_n$$

where $n = 0$ represents the separate slab. Since the terms

$$\sum_{n=-\infty}^{0} 2\pi G\rho ma_n - \sum_{n=1}^{\infty} 2\pi G\rho ma_n$$

are paired one to one as in (3), they cancel each other out and the result is zero. Therefore the total force on m will be $F = 2\pi G\rho ma$, which is an arbitrary value, since a has been arbitrarily chosen.

This new version of the paradox does not use the fact that the potential is null inside a sphere and yet, as in Seeliger's original version, it can return any arbitrary value. It is possible in fact to prove that the paradox occurs with a wide range of forces other than Newton's inverse square law. With Newton's law, the force of each slab is independent of the distance, thus the force exerted by each of the layers is the same and cancels out with another slab located symmetrically from the given particle. However, if we had a different force law in which the gravitational force of each slab were dependent on the distance, we still would be able to repeat the previous calculation by choosing for each slab a suitable thickness so as to exactly balance another slab at the opposite side of the particle, provided that the sum of the forces diverged.

3 Riemann series theorem

In 1827, mathematician Peter Lejeune-Dirichlet discovered the surprising result that some convergent series, when rearranged, can yield a different result [3]. Based on this discovery, another German mathematician, Bernhard Riemann published in 1852 a theorem [3], known today as *Riemann's series theorem* (or *Riemann rearrangement theorem*), proving that in general, infinite series are not associative, that is, they cannot be rearranged.

According to this theorem (see for example [4]), an *absolutely* convergent series will always give the same result, no matter how it is rearranged. However, a *conditionally* convergent series, by a suitable permutation of its elements, can take any arbitrary value or even diverge.

Let's review some definitions. A series converges if there exists a value ℓ such that the sequence of the partial sums

$$\{S_1, S_2, S_3, ...\}\,,\ \text{where } S_n = \sum_{k=1}^{n} a_k$$

converges to ℓ. That is, for any $\epsilon > 0$, there exists an integer N such that if $n \geq N$, then

$$|S_n - \ell| \leq \epsilon\,.$$

A series $S_n = \sum_{n=1}^{\infty} a_n$ converges *absolutely* if $S_n = \sum_{n=1}^{\infty} |a_n|$ converges. A series $S_n = \sum_{n=1}^{\infty} a_n$ converges *conditionally* if it converges but the series $S_n = \sum_{n=1}^{\infty} |a_n|$ diverges.

Riemann's series theorem can be directly extrapolated to conditionally convergent integrals (see for example [5]).

In the case of Seeliger's paradox, we note first that although the masses in the universe should be treated as discrete, Seeliger for simplicity turns them into a homogeneous mass distribution throughout the universe, thus formulating it in terms of integrals instead of series. Like Seeliger, we will work with a continuous mass distribution, but bearing in mind that the problem is actually discrete.

Considering a uniform mass distribution with a volume density ρ, and using a spherical coordinate system (r, θ, ϕ) centered on m, we have that, according to Newton's law, the component of the total force exerted on a particle m along the x axis is

$$F_x = -Gm \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho \sin\phi \cos\phi \, d\phi d\theta dr \,, \qquad (4)$$

and similarly for the other axes.

Since the integral is only conditionally convergent, we have to pay attention to the order in which we calculate the multiple integral. In this case, our goal is to integrate sequentially the shells around the test mass, starting from $r = 0$ and extending to $r = \infty$, thus we have to integrate first over the variables θ and ϕ and only then over r. Note therefore that (4) is not necessarily equal to

$$F_x = -Gm \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} \rho \sin\phi \cos\phi \, dr d\theta d\phi \,.$$

We solve the integral (4)

$$\begin{aligned} F_x &= -Gm \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho \sin\phi \cos\phi \, d\phi d\theta dr \\ &= -Gm \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \rho \left[\sin\left(\frac{-\cos^2\phi}{2} \right) \right]_{\phi=0}^{2\pi} d\theta dr = 0 \,, \end{aligned}$$

which, again, is what could be expected from symmetry. Following Seeliger's procedure, we can calculate the integral in a different way by splitting the space into a sphere of radius a, centered in a point Q separated from m by a distance a, so that the test mass lies on its surface, and concentric shells also centered in Q containing the particle in their interior. In other words, the contribution of every mass in the universe is added but in a different order. Thus the integral is rearranged, which is what Riemanns's theorem warns us against. Taking Q as the origin of coordinates, the x component of the force will be

$$F_x = -\frac{GmM}{a^2} - Gm \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho \sin\phi \cos\phi \, d\phi d\theta dr \,. \quad (5)$$

The first term on the right hand side of (5) is the attraction of the sphere, being M its mass, and a the distance between the particle m and the center of the sphere. The second is the attraction of the concentric shells, which is zero. Therefore,

$$F_x = -\frac{GmM}{a^2} \,.$$

Since the integral is only conditionally convergent, it is no surprise that the new integral obtained by a rearrangement of its terms yields a different result.

Riemann's theorem shows the reason why Seeliger's paradox occurs, and it also demonstrates that its origin is mathematical, not physical.

The integral converges to zero but any other rearrangement of the integral will yield a different value. Given the infinitely many possible results, we are forced to ask which one, if any, is the "correct" value, *i.e.* the one that a measure instrument would register in reality. Riemann's theorem does not provide a way to decide this, having therefore to rely on the physical significance of each reordering of the integral or the series. The following two arguments, although lacking mathematical rigor, both indicate that the only valid way to carry out the calculation is by considering the mass at the center of coordinates:

a) Since all the observable physical magnitudes in this system, *i.e.* the mass distribution, are smooth everywhere, *i.e.* infinitely differentiable (except possibly at the point where the test mass is located), it is required that any derived function be also differentiable. Any discontinuity introduced in any of the magnitudes must be discarded as lacking physical basis. However, the force obtained when we calculate (5) is

$$F(r) = \begin{cases} -\dfrac{4}{3} G\rho\pi M r \,, & r \le R_0 \\ -\dfrac{GMm}{r^2} \,, & r > R_0 \end{cases} \qquad (6)$$

where r is the distance from the test mass to the center of the sphere, and R_0 the radius of the latter. This function is differentiable at $r = R_0$ only if $R_0 = 0$. Thus, the only arrangement of terms which will provide a differentiable force function is the one which considers the test mass at the origin of coordinates.

b) A non-nil result of (5) would be acceptable only if it is a constant finite value independent of r. That would correspond to the whole universe being pushed and moving in one direction with respect to absolute space. Since this absolute space is not detectable, we cannot determine whether this movement is actually taking place or not. However, if the force depends on r, different parts of the universe would be pushed with different forces, giving rise to the motion of some masses with respect to other masses. This is not observed, and thus we have to reject this possibility.

The only case where the force (5) is independent of r is when $F(r) = 0$ everywhere. These two arguments both suggest that the nil result is the only one physically meaningful.

Some authors had already suspected that Seeliger's paradox has no physical relevance, [6], [7], but none of them give a rigorous explanation. It is common to find in the literature regarding Seeliger's paradox, confusing statements about convergence of infinite series [6, 8]. Even Newton, in his famous letter to Bentley [9], erred when he spoke about the stability of an infinite Universe:

> ... if a body stood in equilibrio between any two equal
> and contrary attracting infinite forces, and if to either

> of these forces you add any new finite attracting force, that new force, howsoever little, will destroy their equilibrium.

In the situation described by him we have two opposite infinite sides pulling on each other, or $\infty - \infty$. This is indeterminate and so, it might or might not be stable. However, if we assume the stability of the system, as Newton does, it is obvious that adding a finite quantity of mass to either infinite side will not destroy the equilibrium, since a finite quantity added to an infinite one will not alter the latter, and so it will make no difference in the balance between the two infinite hemispheres of the universe. The universe will thus remain stable.

4 Conclusion

We have proved, with the help of Riemann's series theorem, that Seeliger's paradox has no physical significance. It is the consequence of a flawed manipulation of infinite conditionally convergent integrals. Therefore the paradox cannot be used as a valid argument against Newton's potential or the infiniteness of the universe.

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