On the Incompatibility of the Dirac-like Field Operator with the Majorana Anzatz

Valeriy V. Dvoeglazov

UAF, Universidad Autónoma de Zacatecas Apartado Postal 636, Suc. 3, C. P. 98061, Zacatecas, Zac., México. E-mail: valeri@fisica.uaz.edu.mx

We investigate some subtle points of the Majorana(-like) theories. We show explicitly the incompatibility of the Majorana Anzatz with the Dirac-like field operator in the original Majorana theory in various spin bases.

1 Introduction.

Majorana proposed his theory of neutral particles [1], in fact, on the basis of the Dirac equation [2]. However, the quantum field theory has not yet been completed in 1937. The Dirac equation [2–4] is well known to describe the charged particles. The Dirac-like field operator has not yet been completed in 1937. The Dirac Majorana proposed his theory of neutral particles [1], in fact, in the case of the (1–2) representation. Namely, the QFT vacuum contains up and down spins, (h) annihilations on vacuum.

Their explicit forms and excellent discussion can be found in [7]. However, the action of \( a_h(-p) \equiv a_h(-E_p, -p) \) on the quantum-field vacuum is different (according, in fact, to the consideration below). Namely, the QFT vacuum contains all negative-energy states according to the Dirac interpretation. So when acting \( a_h(-E_p, -p) \) on the vacuum this operator changes it (destroys a “hole”). The result is not zero, as opposed to the action of \( a_h(E_p, p) \) on vacuum.

In general we should transform \( u_h(-p) \) to the \( u(p) \) in order to follow the original Dirac idea, where antiparticles were treated as particles with negative energy. The procedure is the following one [8, 9]. In the Dirac case we should assume the

\[
\Psi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \left[ u_h(p)\hat{u}_h(p)e^{ip\cdot x} + v_h(p)\hat{v}_h(p)e^{-ip\cdot x} \right], \tag{1}
\]
following relation in the field operator:
\[
\sum_h v_h(p) b^+_h(p) = \sum_h u_h(-p) a_h(-p) .
\] (9)

We need \( \Lambda_{\mu l}(p) = \bar{u}_\mu(E_p, \vec{p}) u_l(-E_p, -\vec{p}) \). By direct calculations, we find
\[
-b^+_l(p) = \sum_A \Lambda_{\mu l}(p) u_A(-p) .
\] (10)

where \( \Lambda_{\mu l} = -i(\sigma \cdot n)_{\mu l} \). \( \vec{p} = p/|p| \), and
\[
-b^+_l(p) = +i \sum_A (\sigma \cdot n)_{\mu l} a_A(-p) .
\] (11)

Multiplying (9) by \( \bar{u}_\mu(-E_p, -\vec{p}) \) we obtain
\[
a_{-l}(-p) = -i \sum_A (\sigma \cdot n)_{\mu l} b^+_A(p) .
\] (12)

The equations are self-consistent.

Next, we can introduce the helicity operator of the \((1/2, 0) \oplus (0, 1/2)\) representation:
\[
\hat{h} = \begin{pmatrix} \hat{h} & 0_{2 \times 2} \\ 0_{2 \times 2} & \hat{h} \end{pmatrix} .
\] (13)

where
\[
\hat{h} = \frac{1}{2} (\sigma \cdot \vec{p}) = \frac{1}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i \phi} \\ \sin \theta e^{i \phi} & -\cos \theta \end{pmatrix} .
\] (14)

which commutes with the Dirac Hamiltonian, thus developing the theory in the helicity basis. We can start from the Klein-Gordon equation, generalized for describing the spin-1/2 particles (i.e., two degrees of freedom). Ref. [3]; again \( c = h = 1 \). If the 2-spinors are defined as in [10,11] then we can construct the corresponding \( u^- \) and \( v^- \) 4-spinors in the helicity basis.

\[
u_1 = N^+_1 \left( \frac{\phi_1}{\sqrt{m^2 + \mu^2}} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{m^2 + \mu^2} \\ \sqrt{m^2 + \mu^2} \end{array} \right) ,
\] (15)

\[
u_2 = N^+_1 \left( \frac{\phi_1}{\sqrt{m^2 + \mu^2}} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{m^2 + \mu^2} \\ -\sqrt{m^2 + \mu^2} \end{array} \right) ,
\] (16)

\[
u_3 = N^-_1 \left( \frac{\phi_1}{\sqrt{m^2 + \mu^2}} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{m^2 + \mu^2} \\ -\sqrt{m^2 + \mu^2} \end{array} \right) ,
\] (17)

\[
u_4 = N^-_1 \left( \frac{\phi_1}{\sqrt{m^2 + \mu^2}} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{m^2 + \mu^2} \\ -\sqrt{m^2 + \mu^2} \end{array} \right) ,
\] (18)

where the normalization to the unit was again used. Please note that as in Ref. [14] the \( \gamma^- \) matrices are the same as in the spinorial basis:
\[
\gamma^0 = \begin{pmatrix} 0_{2 \times 2} & 1_{2 \times 2} \\ 1_{2 \times 2} & 0_{2 \times 2} \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0_{2 \times 2} & -\sigma^i \\ \sigma^i & 0_{2 \times 2} \end{pmatrix} .
\] (19)

Thus, in the helicity basis we also have \( v_0(p) = \gamma_5 u_h(p) \) as usual. Next, both \( u^- \) and \( v^- \) spinors above are the eigen spinors of the helicity operator [14] because the 2-spinors \( \phi_h \) are the eigenspinors of \( \hat{h} \).

We again define the field operator as in (2) except for the polarization index \( h \), which now answers for the helicity (not for the third projection of the spin, see [14]). The commutation relations are assumed to be the standard ones [5,6,12,13], except for adjusting the dimensional factor (see the discussion above):
\[
[a_{-h}(p), a^-_{-l}(k)]_+ = 2 E_p \delta^{(3)}(\vec{p} - \vec{k}) \delta_{\mu l} ,
\] (20)

\[
[a_{-h}(p), a^+_l(k)]_+ = 0 = [a^+_l(p), a^-_{-l}(k)]_+ ,
\] (21)

\[
[a_{-h}(p), b^+_l(k)]_+ = 0 = [b^+_l(p), a^-_{-l}(k)]_+ ,
\] (22)

\[
[b_{-h}(p), b^-_{-l}(k)]_+ = 2 E_p \delta^{(3)}(\vec{p} - \vec{k}) \delta_{\mu l} ,
\] (23)

\[
[b_{-h}(p), b^+_l(k)]_+ = 0 = [b^+_l(p), b^-_{-l}(k)]_+ .
\] (24)

However, the attempt is now failed to obtain the previous result (11) for \( \Lambda_{\mu l}(p) \). In this helicity case
\[
\bar{u}_\mu(p) u_{-l}(-p) = i \sigma^\mu_{\mu l} .
\] (25)

Please remember that the changes of the spin bases are performed by the rotation in the spin-parity space.

2 Analysis of the Majorana Anzatz

It is well known that “particle=antiparticle” in the Majorana theory. So, in the language of the quantum field theory we should have
\[
b_{-h}(E_p, \vec{p}) = e^{i \mu^l} a_{-h}(E_p, \vec{p}) .
\] (26)

Usually, different authors use \( \varphi = 0, \pi/2 \) depending on the metrics and on the forms of the 4-spinors and commutation relations. It is related to the Kaysor phase factor.

So, on using (11) and the above-mentioned postulate we come to:
\[
a^+_l(p) = +i e^{i \varphi^l} (\sigma \cdot n)_{\mu l} a_{-l}(-p) .
\] (27)

On the other hand, using (12) we make the substitutions
\[
E_p \rightarrow -E_p, \quad p \rightarrow -\vec{p}
\] to obtain
\[
a_{-h}(p) = +i(\sigma \cdot n)_{\mu l} b^+_l(-p) .
\] (28)

The totally reflected (26) is \( b_{-h}(-E_p, -\vec{p}) = e^{i \mu^l} a_{-h}(-E_p, -\vec{p}) \). Thus,
\[
b^+_l(-p) = e^{-i \mu^l} a^-_{-l}(-p) .
\] (29)

Combining with (28), we come to
\[
a_{-h}(p) = +i e^{-i \varphi^l} (\sigma \cdot n)_{\mu l} a^+_l(p) ,
\] (30)

*However, when discussing the spin properties of \( u(-p) \) and \( v(-p) \) in the helicity basis one should clarify the notational issues. Due to \( \phi_{11}(-p) = -\phi_{11}(p) \), \( u_1(-E_p, -\vec{p}) = \pm u_1(E_p, \vec{p}) \) we have \( \bar{u}_{11}(-E_p, -\vec{p}) = -\frac{1}{2} \bar{u}_{11}(E_p, \vec{p}) \), and similarly for \( s(-p) \) 4-spinors. However, the equation (25) below is valid within the used notation.

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and
\[ a^\dagger_\mu(p) = -ie^{i\Theta}(\sigma^\alpha \cdot n)_\mu a_\mu(-p). \] (31)

This contradicts with the equation (27) unless we have the preferred axis in every inertial system.

Next, we can use another Majorana anzatz \( \Psi = \pm e^{i\alpha}\Psi_c \) with usual definitions
\[ C = \left( \begin{array}{cc} 0 & i\Theta \\ -i\Theta & 0 \end{array} \right) K, \quad \Theta = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) = -i\sigma^y. \] (32)

Thus, on using \( Cu^\dagger_\mu(p) = iv_\lambda(p), Cu^\dagger_\mu(p) = -iv_\lambda(p) \) we come to other relations between creation/annihilation operators
\[ a^\dagger_\mu(p) = \mp ie^{-i\alpha}b^\dagger_\mu(p), \] (33)
\[ a^\dagger_\mu(p) = \mp ie^{-i\alpha}b^\dagger_\mu(p), \] (34)

which may be used instead of (26). Due to the possible signs \( \pm \) the number of the corresponding states is the same as in the Dirac case that permits us to have the complete system of the Fock states over the \( (1/2, 0) \oplus (0, 1/2) \) representation space in the mathematical sense.\(^{\ast}\) However, in this case we deal with the self/anti-self charge conjugate quantum field operator instead of the self/anti-self charge conjugate quantum states. Please remember that it is the latter that answers for neutral particles; the quantum field operator contains the information neutral or charged.

As a discussion we observe that the origins and the consequences of the contradiction between (27) and (31) may be the following. In general, the QFT space reflection are performed by the unitary transformations in the Fock space. The time reflection is performed by the anti-unitary transformation. However, after writing the present paper I learnt from [15] about arguments of unitary time reversal on the first quantization level. What would be the influence of this proposition on the second quantization scheme and on the Majorana anzatz should be the subject of future publications.

3 Conclusions

We conclude that something is missed in the foundations of the original Majorana theory and/or the Dirac “hole” theory. At the moment the above consideration points to the rotational symmetry breaking after application of the Majorana anzatz in the \( (1/2, 0) \oplus (0, 1/2) \) representation, for higher spins as well [16].

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\(^{\ast}\)Please note that the phase factors may have physical significance in quantum field theories as opposed to the textbook nonrelativistic quantum mechanics, as was discussed recently by several authors.

References