On the Incompatibility of the Dirac-like Field Operator with the Majorana Anzatz

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We investigate some subtle points of the Majorana(-like) theories. We show explicitly the incompatibility of the Majorana Anzatz with the Dirac-like field operator in the original Majorana theory in various spin bases.

1 Introduction.

Majorana proposed his theory of neutral particles [1], in fact, on the basis of the Dirac equation [2]. However, the quantum field theory has not yet been completed in 1937. The Dirac equation [2–4] is well known to describe the charged particles of the spin 1/2.

Usually, everybody uses the following definition of the field operator [5]:

$$\Psi(x) = \frac{1}{(2\pi)^3}\sum_h \int \frac{d^3\mathbf{p}}{2E_p}\Big[u_h(\mathbf{p})a_h(\mathbf{p})e^{-ip\cdot x} + v_h(\mathbf{p})b_h^\dagger(\mathbf{p})e^{+ip\cdot x}\Big], \quad (1)$$

as given *ab initio*. After actions of the Dirac operator at $\exp(\mp i p_\mu x^\mu)$ the 4-spinors ($u-$ and $v-$) satisfy the momentum-space equations: $(\hat{p} - m)u_h(p) = 0$ and $(\hat{p} + m)v_h(p) = 0$, respectively; the h is the polarization index; $\hat{p} = p^\alpha\gamma_\alpha$. It is easy to prove from the characteristic equations $Det(\hat{p} \mp m) = (p_0^2 - \mathbf{p}^2 - m^2)^2 = 0$ that the solutions should satisfy the energy-momentum relation $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$ with both signs of p_0.

However, the general method of construction of the field operator has been given in the Bogoliubov and Shirkov book [6]. In the case of the $(1/2, 0) \oplus (0, 1/2)$ representation we have:

$$\Psi(x) = \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2 - m^2)e^{-ip\cdot x}\Psi(p) = \frac{\sqrt{m}}{(2\pi)^3}\sum_{h=\pm 1/2}\int \frac{d^3\mathbf{p}}{2E_p}\theta(p_0)\Big[u_h(p)a_h(p)|_{p_0=E_p}e^{-i(E_pt-\mathbf{p}\cdot\mathbf{x})} + u_h(-p)a_h(-p)|_{p_0=E_p}e^{+i(E_pt-\mathbf{p}\cdot\mathbf{x})}\Big]. \quad (2)$$

$\theta(p_0)$ is the Heaviside function(al). During these calculations we did not yet assume, which equation did this field operator (namely, the $u-$ spinor) satisfy (apart from the Klein-Gordon equation), with negative- or positive- mass. The explicit introduction of the factor $\sqrt{m}$ is caused by the following consideration. The 4-spinor normalization is known [4] to be able being chosen to the unit:

$$\bar{u}_{(\mu)}(p)u_{(\lambda)}(p) = +\delta_{\mu\lambda}\,, \quad (3)$$

$$\bar{u}_{(\mu)}(p)u_{(\lambda)}(-p) = 0\,, \quad (4)$$

$$\bar{v}_{(\mu)}(p)v_{(\lambda)}(p) = -\delta_{\mu\lambda}\,, \quad (5)$$

$$\bar{v}_{(\mu)}(p)u_{(\lambda)}(p) = 0\,, \quad (6)$$

where μ and λ are the polarization indices. The action should be dimensionless in $c = \hbar = 1$. Thus, the Lagrangian density has the dimension $[energy]^4$, and the 4-spinor field, the dimension $[energy]^{3/2}$. From (3-6) we see that the momentum-space 4-spinors should be dimensionless in this formulation. The creation/ahhihilation operators should have the dimension $[energy]^{-1}$ if we want to keep the standard (anti) commutation relations (20-24). Therefore, a factor with the dimension $[energy]^{1/2}$ can be introduced explicitly in (2) for the sake of conveniency instead of that in the normalizations or in the anticommutation relations [5].

The creation/annihilation quantum-field operators are defined by their actions on the quantum-field states in the representation of the occupation numbers:

$$a_h^\dagger(E_p, \mathbf{p})|n>=|n+1; \mathbf{p}, h>, \quad a_h(E_p, \mathbf{p})|n>=|n-1; \mathbf{p}, h>, \quad (7)$$

$$a_h(E_p, \mathbf{p})|0>= 0\,. \quad (8)$$

Their explicit forms and excellent discussion can be found in [7]. However, the action of $a_h(-p) \equiv a_h(-E_p, -\mathbf{p})$ on the quantum-field vacuum is different (according, in fact, to the consideration below). Namely, the QFT vacuum contains all negative-energy states according to the Dirac interpretation. So when acting $a_h(-E_p, -\mathbf{p})$ on the vacuum this operator changes it (destroys a "hole"). The result is *not* zero, as opposed to the action of $a_h(+E_p, \mathbf{p})$ on vacuum.*

In general we should transform $u_h(-p)$ to the $v(p)$ in order to follow the original Dirac idea, where antiparticles were treated as particles with negative energy. The procedure is the following one [8, 9]. In the Dirac case we should assume the

*The similar situation is encountered in quantum mechanics of harmonic oscillator, where the creation operator can be obtained after application of reflection operators to the annihilation operator, and vice versa. This is not surprising because quantum field theory has the oscillator representation too.

following relation in the field operator:

$$\sum_h v_h(p) b_h^\dagger(p) = \sum_h u_h(-p) a_h(-p)\,. \tag{9}$$

We need $\Lambda_{\mu\lambda}(\mathbf{p}) = \bar{v}_\mu(E_p, \mathbf{p}) u_\lambda(-E_p, -\mathbf{p})$. By direct calculations, we find

$$-b_\mu^\dagger(p) = \sum_\lambda \Lambda_{\mu\lambda}(p) a_\lambda(-p)\,. \tag{10}$$

where $\Lambda_{\mu\lambda} = -i(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda}$, $\mathbf{n} \equiv \widehat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$, and

$$b_\mu^\dagger(p) = +i \sum_\lambda (\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} a_\lambda(-p)\,. \tag{11}$$

Multiplying (9) by $\bar{u}_\mu(-E_p, -\mathbf{p})$ we obtain

$$a_\mu(-p) = -i \sum_\lambda (\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} b_\lambda^\dagger(p)\,. \tag{12}$$

The equations are self-consistent.

Next, we can introduce the helicity operator of the $(1/2,0)\oplus(0,1/2)$ representation:

$$\widehat{h} = \begin{pmatrix} \hat{h} & 0_{2\times2} \\ 0_{2\times2} & \hat{h} \end{pmatrix}. \tag{13}$$

where

$$\hat{h} = \frac{1}{2}\boldsymbol{\sigma}\cdot\widehat{\mathbf{p}} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{+i\phi} & -\cos\theta \end{pmatrix}, \tag{14}$$

which commutes with the Dirac Hamiltonian, thus developing the theory in the helicity basis. We can start from the Klein-Gordon equation, generalized for describing the spin-1/2 particles (i. e., two degrees of freedom), Ref. [3]; again $c = \hbar = 1$. If the 2-spinors are defined as in [10, 11] then we can construct the corresponding $u-$ and $v-$ 4-spinors in the helicity basis.

$$u_\uparrow = N_\uparrow^+ \begin{pmatrix} \phi_\uparrow \\ \frac{E-p}{m}\phi_\uparrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{E+p}{m}}\phi_\uparrow \\ \sqrt{\frac{m}{E+p}}\phi_\uparrow \end{pmatrix}, \tag{15}$$

$$u_\downarrow = N_\downarrow^+ \begin{pmatrix} \phi_\downarrow \\ \frac{E+p}{m}\phi_\downarrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{m}{E+p}}\phi_\downarrow \\ \sqrt{\frac{E+p}{m}}\phi_\downarrow \end{pmatrix}, \tag{16}$$

$$v_\uparrow = N_\uparrow^- \begin{pmatrix} \phi_\uparrow \\ -\frac{E-p}{m}\phi_\uparrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{E+p}{m}}\phi_\uparrow \\ -\sqrt{\frac{m}{E+p}}\phi_\uparrow \end{pmatrix}, \tag{17}$$

$$v_\downarrow = N_\downarrow^- \begin{pmatrix} \phi_\downarrow \\ -\frac{E+p}{m}\phi_\downarrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{m}{E+p}}\phi_\downarrow \\ -\sqrt{\frac{E+p}{m}}\phi_\downarrow \end{pmatrix}, \tag{18}$$

where the normalization to the unit was again used. Please note that as in Ref. [14] the $\gamma-$ matrices are the same as in the spinorial basis:

$$\gamma^0 = \begin{pmatrix} 0_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 0_{2\times2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0_{2\times2} & -\boldsymbol{\sigma}^i \\ \boldsymbol{\sigma}^i & 0_{2\times2} \end{pmatrix}. \tag{19}$$

Thus, in the helicity basis we also have $v_h(p) = \gamma_5 u_h(p)$ as usual. Next, both $u-$ and $v-$ spinors above are the eigenspinors of the helicity operator [14] because the 2-spinors ϕ_h are the eigenspinors of $\hat{h}$.*

We again define the field operator as in (2) except for the polarization index h, which now answers for the helicity (not for the third projection of the spin, see [14]). The commutation relations are assumed to be the standard ones [5,6,12,13], except for adjusting the dimensional factor(see the discussion above):

$$\left[a_\mu(\mathbf{p}), a_\lambda^\dagger(\mathbf{k})\right]_+ = 2E_p \delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{\mu\lambda}\,, \tag{20}$$

$$\left[a_\mu(\mathbf{p}), a_\lambda(\mathbf{k})\right]_+ = 0 = \left[a_\mu^\dagger(\mathbf{p}), a_\lambda^\dagger(\mathbf{k})\right]_+\,, \tag{21}$$

$$\left[a_\mu(\mathbf{p}), b_\lambda^\dagger(\mathbf{k})\right]_+ = 0 = \left[b_\mu(\mathbf{p}), a_\lambda^\dagger(\mathbf{k})\right]_+\,, \tag{22}$$

$$\left[b_\mu(\mathbf{p}), b_\lambda^\dagger(\mathbf{k})\right]_+ = 2E_p \delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{\mu\lambda}\,, \tag{23}$$

$$\left[b_\mu(\mathbf{p}), b_\lambda(\mathbf{k})\right]_+ = 0 = \left[b_\mu^\dagger(\mathbf{p}), b_\lambda^\dagger(\mathbf{k})\right]_+\,. \tag{24}$$

However, the attempt is now failed to obtain the previous result (11) for $\Lambda_{\mu\lambda}(p)$. In this helicity case

$$\bar{v}_\mu(p) u_\lambda(-p) = i\sigma^y_{\mu\lambda}\,. \tag{25}$$

Please remember that the changes of the spin bases are performed by the rotation in the spin-parity space.

2 Analysis of the Majorana Anzatz

It is well known that "*particle=antiparticle*" in the Majorana theory. So, in the language of the quantum field theory we should have

$$b_\mu(E_p, \mathbf{p}) = e^{i\varphi} a_\mu(E_p, \mathbf{p})\,. \tag{26}$$

Usually, different authors use $\varphi = 0, \pm\pi/2$ depending on the metrics and on the forms of the 4-spinors and commutation relations. It is related to the Kayser phase factor.

So, on using (11) and the above-mentioned postulate we come to:

$$a_\mu^\dagger(p) = +ie^{i\varphi}(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} a_\lambda(-p)\,. \tag{27}$$

On the other hand, on using (12) we make the substitutions $E_p \to -E_p$, $\mathbf{p} \to -\mathbf{p}$ to obtain

$$a_\mu(p) = +i(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} b_\lambda^\dagger(-p)\,. \tag{28}$$

The totally reflected (26) is $b_\mu(-E_p, -\mathbf{p}) = e^{i\varphi} a_\mu(-E_p, -\mathbf{p})$. Thus,

$$b_\mu^\dagger(-p) = e^{-i\varphi} a_\mu^\dagger(-p)\,. \tag{29}$$

Combining with (28), we come to

$$a_\mu(p) = +ie^{-i\varphi}(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} a_\lambda^\dagger(-p)\,, \tag{30}$$

*However, when discussing the spin properties of $u(-p)$ and $v(-p)$ in the helicity basis one should clarify the notational issues. Due to $\phi_{\uparrow\downarrow}(-\mathbf{p}) = -i\phi_{\downarrow\uparrow}(\mathbf{p})$, $u_{\uparrow\downarrow}(-E_p, -\mathbf{p}) = \pm v_{\downarrow\uparrow}(E_p, \mathbf{p})$ we have $\widehat{h} u_{\uparrow\downarrow}(-E_p, -\mathbf{p}) = -\frac{1}{2} v_{\downarrow\uparrow}(E_p, \mathbf{p})$, and similarly for $v(-p)$ 4-spinors. However, the equation (25) below is valid within the used notation.

and

$$a^{\dagger}_{\mu}(p) = -ie^{i\varphi}(\boldsymbol{\sigma}^* \cdot \mathbf{n})_{\mu\lambda} a_\lambda(-p)\,. \tag{31}$$

This contradicts with the equation (27) unless we have the preferred axis in every inertial system.

Next, we can use another Majorana anzatz $\Psi = \pm e^{i\alpha}\Psi^c$ with usual definitions

$$C = \begin{pmatrix} 0 & i\Theta \\ -i\Theta & 0 \end{pmatrix}\mathcal{K}\,, \quad \Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma^y\,. \tag{32}$$

Thus, on using $Cu^*_\uparrow(\mathbf{p}) = iv_\downarrow(\mathbf{p})$, $Cu^*_\downarrow(\mathbf{p}) = -iv_\uparrow(\mathbf{p})$ we come to other relations between creation/annihilation operators

$$a^{\dagger}_{\uparrow}(\mathbf{p}) = \mp ie^{-i\alpha} b^{\dagger}_{\downarrow}(\mathbf{p})\,, \tag{33}$$

$$a^{\dagger}_{\downarrow}(\mathbf{p}) = \pm ie^{-i\alpha} b^{\dagger}_{\uparrow}(\mathbf{p})\,, \tag{34}$$

which may be used instead of (26). Due to the possible signs $\pm$ the number of the corresponding states is the same as in the Dirac case that permits us to have the complete system of the Fock states over the $(1/2, 0)\oplus(0, 1/2)$ representation space in the mathematical sense.* However, in this case we deal with the self/anti-self charge conjugate quantum field operator instead of the self/anti-self charge conjugate quantum states. Please remember that it is the latter that answers for neutral particles; the quantum field operator contains the information about more than one state, which may be either electrically neutral or charged.

As a discussion we observe that the origins and the consequences of the contradiction between (27) and (31) may be the following. In general, the QFT space reflection are performed by the unitary transformations in the Fock space. The time reflection is performed by the anti-unitary transformation. However, after writing the present paper I learnt from [15] about arguments of unitary time reversal on the first quantization level. What would be the influence of this proposition on the second quantization scheme and on the Majorana Anzatz should be the subject of future publications.

3 Conclusions

We conclude that something is missed in the foundations of the original Majorana theory and/or the Dirac "hole" theory. At the moment the above consideration points to the rotational symmetry breaking after application of the Majorana Anzatz in the $(1/2, 0) \oplus (0, 1/2)$ representation, for higher spins as well [16].

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*Please note that the phase factors may have physical significance in quantum field theories as opposed to the textbook nonrelativistic quantum mechanics, as was discussed recently by several authors.

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