Unified Two Dimensional Spacetime for the River Model of Gravity and Cosmology

Alexander Kritov

E-mail: alex@kritov.ru

Within the proposed assumptions, including the existence of the discrete (minimally uncertain) volume of space, the possibility of mapping of Euclidean 3D to 1D space in the spherically symmetric case is considered. In introduced unified pseudo-Minkowski 2D spacetime (t,η) the river velocity for the Schwarzschild metric represents the uniform acceleration. The Rindler coordinate transforms in 2D spacetime lead to the Schwarzschild-de Sitter metric in static 4D coordinates and result in the scale factor that coincides with the one for cosmological expansion for the Universe with dark energy. The FLRW metric with such scale factor has the conformal form in unified 2D spacetime, and the varying Hubble parameter can be expressed with conformal time via the simple expression. The dynamic and continuity of the uniformly accelerated Rindler flow in unified 2D spacetime are reviewed.

The river model of gravity and the analog gravity is an alternative to the General Relativity (GR) approach to gravitation. The purpose of this article is to exhibit the analogy between the radial river velocity in three spatial dimensions with the motion along one spatial dimension. In the beginning, the three new physical parameters are to be introduced: the mass-radius, the discrete volume of space, and the new spatial coordinate η that is mapped to three spatial dimensions which allows introducing unified two-dimensional space-time (t, η). Note: Only the case of spherical symmetry is reviewed.

1 The river model of gravity and the equivalence principle

The river model of gravity [5] and the analog gravity [2] is the approach to gravity where the equivalence principle (EP) holds. But it is interpreted in such a way that instead of equivalence of gravity to the acceleration, it aligns gravity with non-uniform velocity $v(r)$ denoted as the river velocity. In the analog gravity models, the velocity $v(r)$ is considered to be a movement of some physical medium in flat background spacetime. The flow of the medium is considered to be stationary and irrotational. The use of non-uniform $v(r)$ instead of the acceleration provides the intuitively obvious connection to the metric in static coordinates

$$ds^2 = -c^2\left(1 - \frac{v^2}{c^2}\right)dt'^2 + \left(1 - \frac{v^2}{c^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (1)$$

where $d\Omega^2 = \sin^2\theta d\phi^2 + d\theta^2$ and coordinate time is denoted as t'. Contrary to that, attempts to embed the acceleration from the EP to a similar form of the metric are still highly disputable.

It was demonstrated in [8] using the coordinate transforms that the static metric (1) in the comoving reference frame has the following *equivalent* form

$$ds^2 = -c^2 d\tau^2 + a(\tau)^2\left(dR^2 + R^2 d\Omega^2\right) \quad (2)$$

which is the Robertson-Walker (FLRW) metric for the spatially flat case ($k = 0$) and $a(\tau)$ is the scale factor related to the river velocity as $v = R\dot{a}$, and v is the proper velocity of the comoving frame. Such equivalency of the static metric (1) to (2) is known for the de Sitter metric only (for example [16]), and the river velocity is associated with the Hubble flow. But the conformity between an arbitrary static metric and the comoving metric (2) in general case is missing or avoided in the literature. Recently, however, Mitra [10] proposed the clarifying view on this problem, which supports the presented approach.

2 The prerequisites of the model

Three postulates of the model are

1. The fundamental significance of the Hubble constant H_0*. The term "varying Hubble constant" can be misleading and is not applied to the approach. The constant is the fundamental value that does not vary with time. Instead it is proposed to use the varying parameter $\mathcal{H}(\tau) = \dot{a}/a$. The significance of it is distinguished from the Hubble constant. Further, the Hubble constant H_0 is denoted as H for shortness.

2. The incompressibility of the fluid and its constant density. It was given in [7], based on the conformal factor issue in the analog gravity and on the continuity equation. The significance of the moving fluid and moving space is the same in the presented approach which allows having aether overtones in the interpretation of such models.

3. The outward direction of the fluid from the center of mass. Czerniawski [4] pointed out that the Gullstrand-Painlevé metric can be written with negative and positive v equivalently. The same is given in [7, 8] for the analog gravity based on the fact that the river velocity comes to the static metric as squared value. If the river velocity depends on central mass then it hardly can be modeled by ingoing flow as the flow at a

*As an example, Dirac's large number coincidence can indirectly support this point or as it was conjectured in [9] $H_0 = m_e c^2/(2^{128}\hbar)$.

distance r somehow should "know" the value of mass located at the point $r = 0$, which intuitively would contradict to the sense of the short-range action of the hydrodynamics.

3 Mass-radius r_m and mass-volume V_m

Let m be a point mass of an elementary particle in the center of a sphere with radius r. Let's designate the certain radius r_m of the spherical volume V_m such as

$$m = \rho_0 \left(\frac{4}{3}\pi r_m^3\right) \qquad \rho_0 = k\rho_c \tag{3}$$

denoting them respectively as mass-radius and the mass-volume. The value of the fluid density ρ_0 is expressed via the critical density ρ_c and k is some coefficient of order of unity and its estimates are given later. Then it can be also noted that

$$r_m = \left(\frac{3}{4\pi}\frac{m}{\rho_0}\right)^{1/3} = \left(\frac{2Gm}{kH^2}\right)^{1/3} . \tag{4}$$

As an example, for the river velocity in case of the Schwarzschild gravity [3, 5]

$$\upsilon(r) = \sqrt{\frac{2Gm}{r}} \tag{5}$$

the equation motion of a fluid (directed outwards as postulated) can be simplified as

$$r(t) = \left(\frac{3}{2}\sqrt{2Gm}\, t\right)^{2/3} = k^{1/3} r_m \left(\frac{3}{2}\, Ht\right)^{2/3} . \tag{6}$$

In such case the space is expanding in outwards direction and its spherical volume within the radius r denoted further as V increases with time as

$$V(t) = V_m k \left(\frac{3}{2}\, Ht\right)^2 \tag{7}$$

near the mass m. The definition of comoving distance R is $r = Ra$. Then one can note that particularly the scale factor can be represented as

$$r(t) = r_m k^{1/3} a(t) \qquad a(t) = \left[\frac{V(t)}{kV_m}\right]^{1/3} . \tag{8}$$

Importantly, the scale factor defined in such does not depend on the value of point mass. The reviewed case yields

$$a(t) = \left(\frac{3}{2}\, Ht\right)^{2/3} . \tag{9}$$

The expression describes the scale factor near the point mass m, for example, near the elementary particle that implies the spatial flow with river velocity (5) corresponding to the Schwarzschild space-time geometry. Further, it will be referred as the scale factor if one may still assume that it just coincidences with the cosmological scale factor.

4 The discrete volume of space V_0

The second parameter that has to be introduces is the minimal measurable volume of space V_0, the constant such as

$$V_0 = \frac{m_0}{\rho_0} \tag{10}$$

where m_0 is minimal mass quanta that is defined as

$$m_0 = \frac{\hbar}{c^2}\beta H \tag{11}$$

based on the uncertainty relation and where β is some coefficient of order of unity, which is determined later*. The existence of such volume implies the uncertainty to measure simultaneously three spatial coordinates as

$$\Delta x\, \Delta y\, \Delta y \geq V_0 \,. \tag{12}$$

The existence of a discrete value for the volume of space can be conjectured as its fundamental property. As the Heisenberg uncertainty principle governs the linear 1D coordinate measurement, the minimal 2D area that corresponds to one bit of the information is the Planck area, then V_0 represents 3D the volume of space with minimal entropy or unit of information that can be measured. The substitution of the value for ρ_0 into (10) leads to

$$V_0 = \left(\frac{2\beta}{3k}\frac{c}{H}\right) S_{Pl} \tag{13}$$

where S_{Pl} is the Planck area. In order to evaluate the volume V_0 as sphere the large number relations from [9, the expressions (1) and (2.3)] can be applied to obtain exactly

$$V_0 = \frac{4\pi}{3}\left(\frac{\beta}{k}\right) r_e \lambda_e \lambda_p \tag{14}$$

where λ_p and λ_e are the de Broglie wavelength of proton and electron and r_e is the classical electron radius†. Notably, the expression shows that V_0 can be expressed via the properties of fundamental particles and λ_p with the dimensionless coefficients, which are determined later.

The minimal volume V_0 can also signify one bit of information as in terms of the total entropy of the Universe within the Hubble volume as substitution leads to

$$I = \frac{V_H}{V_0} = \left(\frac{k}{2\beta}\right)\frac{S_H}{S_{Pl}} \tag{15}$$

where S_H is the area of the Hubble horizon, and the second equality represents the Holographic principle, which should have some the numerical factor here as the identity on the left-hand side represents the entropy of pure space only (without matter and energy). The expression to be used further for V_m via V_0 obviously can be obtained as

$$V_m = V_0\,\frac{m}{m_0} = \frac{V_0}{\lambda_m}\,\frac{c}{\beta H} \tag{16}$$

where λ_m is the de Broglie wavelength of the mass m.

*So V_0 can be simply treated as the mass-volume of m_0.

†with factor of 3/10, as per cited work.

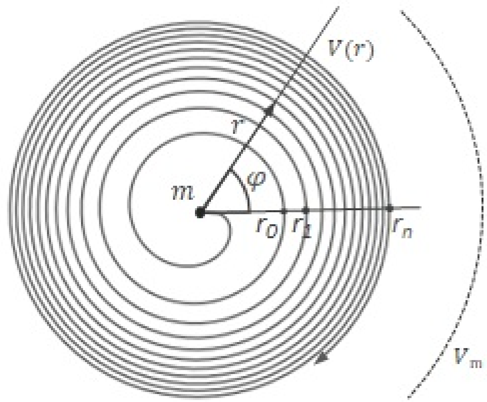

Fig. 1: The mapping of the spherical volume $V(r)$ to unified coordinate where $\eta = \phi\lambda_m/4\pi$ is represented by the angle ϕ. The spiral is given by the polar equation $r = a\phi^{1/3}$. Every turnover cycle corresponds to $dV = V_0$ and to the line segment with length λ_m in η coordinate.*

5 The unified coordinate η

The existence of discrete volumes leads to the proposition that 3D manifold may represent a countable set. Therefore all such V_0's within some spherical volume $V(r)$ can be mapped to fixed-line segments of one-dimensional coordinate. However, as V_0 is the quantity but is not an actual shape; therefore, such mapping is not uniquely defined. The new spatial-like coordinate η can be introduced† as following

$$\vec{\eta} = \lambda_m \frac{V(r)}{V_0} \vec{e_\eta} \,. \tag{17}$$

Such representation provides the mapping of the linear uncertainty relation for λ_m to the uncertainty for 3D volume V_0. The appearance of λ_m in the definition of η is motivated by its presence in (14), implying its fundamental significance as one of V_0's dimension. The coordinate can be understood as constituted of numbers of discrete deltas with the length of λ_m. Each of these deltas corresponds to *next in raw* V_0 within the spherical volume of $V(r)$.

The coordinate transformation likely represents the non-conformal mapping as it all angular information (ϕ,θ) of coordinates in 3D is lost as uses radial distance only. On another hand, the spherical shell with the volume $V_0 = 4\pi r^2 dr$ already does not have angular information due to the uncertainty of V_0. In such a way, the transformation is conformal. The definition can be also written in terms of differentials as

$$d\eta = dV \frac{\lambda_m}{V_0} \,. \tag{18}$$

*The spiral shows resemblance to the Theodorus spiral but constructed with the cubic roots instead of the square roots as $r_n = r_0[n^{1/3} - (n-1)^{1/3}]$.

†It can also be associated with the mass of space in spherical volume with postulated uniform density.

The ratio dV/V_0 corresponds to the natural number n (which is the number of spiral cycles as depicted in Fig. 1). In case if $V(r)$ as is not constant or there is a non-zero flux of the fluid, then it corresponds to the velocity

$$u = \frac{\partial\eta}{\partial t} = \frac{\lambda_m}{V_0}\left(\frac{\partial V}{\partial t}\right). \tag{19}$$

The equation provides the direct correspondence between fluid flow in three-dimensional space and the velocity along the unified coordinate η. Then for the spherically symmetric case, the radial river velocity can be obtained as

$$\upsilon = \frac{V_0}{\lambda_m}\frac{u}{4\pi r^2} \,. \tag{20}$$

The meaning of the expression is evident with the help of Fig. 1, where the velocity u is angular velocity along the spiral line, and υ is its projection to the radial direction. Substitution of (16) leads to

$$\upsilon = V_m \frac{\beta H}{c}\frac{u}{4\pi r^2} \,. \tag{21}$$

Also, the substitution of (16) into (17) provides the spherical volume expressed via η as

$$V = \eta\, V_m \frac{\beta H}{c} \,. \tag{22}$$

Noting the special point on η coordinate

$$\eta_m = \frac{c}{\beta H} \tag{23}$$

that corresponds to mass-radius r_m in 4D spacetime.

6 The motion along η in non-relativistic approximation

With the use of introduced coordinate, the space flow (7) can be represented as an equation of motion along η. The equation (19) for the Schwarzschild case above (7) (differentiating it with respect to time) gives

$$u = \frac{\lambda_m}{V_0}\left(V_m k \frac{9}{2} H^2 t\right). \tag{24}$$

Applying (16)

$$u = \left(\frac{9k}{2\beta} Hc\right) t \tag{25}$$

which is the accelerated motion along coordinate η with constant acceleration‡

$$\alpha = \frac{9k}{2\beta} Hc \,. \tag{26}$$

Those, the Schwarzschild gravity with the river velocity (5) and for the scale factor $a(t)$ as in (9) represent *non-relativistic approximation* of motion with the constant acceleration (26) along coordinate η when $u \ll c$ or at near field of the point mass.

‡In the author's previous work [7] it was assumed that $k = 1$ and $\beta = \frac{3}{2}$ leading to $\alpha = 3Hc$ and (16) corresponds to the volume conversion relation.

7 The relativistic motion along η

It has to be considered now that unified coordinate η belongs to two dimensional Minkowski spacetime with the invariant line element

$$ds^2 = -c^2 dt^2 + d\eta^2 \ . \qquad (27)$$

The relativistic motion with the constant proper acceleration corresponds to the Rindler or also known as Kottler-Møller coordinates transforms [12, 13]

$$t = \frac{c}{\alpha}\sinh\left(\frac{\alpha}{c}\tau\right) \qquad (28)$$

where τ is proper time and t is coordinate time and α is given by (26). The two-velocity is

$$u^i = c\left(\cosh\left(\frac{\alpha}{c}\tau\right), \sinh\left(\frac{\alpha}{c}\tau\right)\right) \qquad (29)$$

where $i = 0, 1$. And the equation of motion along the coordinate is

$$\eta = \eta_0 \cosh\left(\frac{\alpha}{c}\tau\right) - \eta_0 \qquad (30)$$

where the initial conditions are set in such way that $\eta = 0$ at $t = 0$ (because of $V(0) = 0$ as (22)) and the Rindler horizon distance is

$$\eta_0 = \frac{c^2}{\alpha} = \left(\frac{2\beta}{9k}\right)\frac{c}{H} \ . \qquad (31)$$

The significance of such distance is the fact that the moving object can not receive any information from the point of its origin anymore. Therefore, the dependency of gravitation from central mass should vanish*. The substitution of the equation of motion via η (30) to expression for spherical volume (22) leads to

$$V(\tau) = V_m \frac{\beta H c}{\alpha}\left[\cosh\left(\frac{\alpha}{c}\tau\right) - 1\right] . \qquad (32)$$

Expressing the hyperbolic cosine via half of argument of hyperbolic sine and using (8) the scale factor is

$$a(\tau) = \left(\frac{2\beta H c}{k\alpha}\right)^{1/3}\left[\sinh\left(\frac{\alpha}{2c}\tau\right)\right]^{2/3} \qquad (33)$$

where expression for α can be easily substituted from (26). The substitution of the proper velocity u^1 from (29) into (21), expressing the hyperbolic sine by the hyperbolic cosine from (32) with the use of $kr_m^3 H^2 = 2Gm$ (4) lead to the solution for the radial river velocity for spherically symmetric gravitational field of point mass

$$\upsilon(r) = \left(\left[\frac{2\beta}{3k}\frac{\alpha}{3Hc}\right]\frac{2Gm}{r} + \left[\frac{\alpha}{3Hc}\right]^2 H^2 r^2\right)^{1/2} \qquad (34)$$

which is the river velocity for the Schwarzschild-de Sitter (SdS) metric with the additional repulsive Λ-term.

*Starting from this distance the de Sitter model has to be valid, see Section 9.

The scale factor (33) coincidences with the one used in the standard cosmology for the current "dark energy dominated" epoch where it has the following form (see for example [15])

$$a(\tau) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}\left[\sinh\left(\sqrt{\Omega_\Lambda}\frac{3}{2}H\tau\right)\right]^{2/3} . \qquad (35)$$

Matching the Ω's parameters with obtained result (33) leads to

$$\Omega_m = \left[\frac{2\beta}{3k}\frac{\alpha}{3Hc}\right] \qquad \Omega_\Lambda = \left[\frac{\alpha}{3Hc}\right]^2 . \qquad (36)$$

Comparing this with two factors multiplying respectively the first and the second term in the expression (34) one can see that they are surprisingly *identical*.

The presented approach, however, attaches the different significance to these coefficients. The first one implies how the Newtonian gravity deviates from its usual law by simply multiplying the Newtonian potential. It should be set to unity, therefore, which is the condition explicitly equivalent to setting up the value of the acceleration α to (26). Then setting the first parameter to unity and the substitution of the value for α from (26)

$$\upsilon(r) = \left(\frac{2Gm}{r} + \left[\frac{3k}{2\beta}\right]^2 H^2 r^2\right)^{1/2} . \qquad (37)$$

The second factor signifies how repulsive Λ-term differs from $(H^2 r^2)$, and it also consequently adds the pre-factor for H in the de Sitter metric and multiplies the cosmological horizon c/H with the same value (see also (13)).

Further, in the frame of this model, the second parameter is set to unity which equivalently implies the following

$$\frac{3k}{2\beta} = 1 \qquad \alpha = 3Hc \qquad (38)$$

and the pre-factor in the expression for the scale factor (33) becomes unity. In such case, the Rindler horizon (23) as the radial distance from the center of mass

$$r_R = r_m\left(\frac{\beta}{3}\right)^{1/3} \qquad (39)$$

and the distance where the SdS river velocity (37) as function of r approaches its minimum†

$$r(\upsilon_{min}) = r_m\left(\frac{k}{2}\right)^{1/3} \qquad (40)$$

are both coincidences. The possible case can be considered if one also equates the Rindler horizon distance η_0 (23) with η_m (31) then it would lead to $\beta = 3$ and $k = 2$ then the both expressions above would have no prefactors.

†Equating the derivative to zero and using $kr_m^3 H^2 = 2Gm$ as per (4). Another two extreme points of $\upsilon(r)$ where it approaches c are given in [6].

The substantial fact that the Rindler transforms in unified 2D spacetime of the form (28) results in the switch from the Schwarzschild river velocity to the SdS gravity with the repulsive Λ-term in 4D spacetime, by taking into account the relativistic consideration for the uniform acceleration along η. Importantly the obtained river velocity for the SdS metric corresponds to the proper velocity of u^1 in unified spacetime and rationale for it is given in Section 10.

8 The FLRW metric in 2D and the conformal form

As it was done for 4D in Section 2 the scale factor a' for 2D spacetime can be introduced in the same way as

$$\eta = k\,\eta_m\, a'(\tau)\,. \tag{41}$$

Using (30), (31) and (23) with determined coefficients (38) results in

$$a'(\tau) = \sinh^2\left(\frac{3}{2}\,H\tau\right) \tag{42}$$

that corresponds to the following 2D metric

$$ds^2 = -c^2 d\tau^2 + \left[\sinh\left(\frac{3}{2}\,H\tau\right)\right]^4 dz^2 \tag{43}$$

where z is the comoving distance, $u^1 = z\,\dot{a}'$ and τ is the proper time in the comoving frame*. Such form is the mapping of the Robertson Walker (FLRW) metric with the scale factor (33) to 2D spacetime. The metric is written for the fluid while it moves in pseudo-Minkowski spacetime (27). Contrary to the FLRW metric with the scale factor (33), (35) this metric has the conformal form. The conformal time τ' such as $d\tau = d\tau' a'(\tau)$ is given by the transform

$$\tau' = \int \frac{d\tau}{a'(\tau)} = -\frac{2}{3H\tanh\left(\frac{3}{2}\,H\tau\right)} \tag{44}$$

where the integration constant can be set to zero. Notably, conformal time has reversed direction opposite to τ

$$\tau' \in \left(-\infty, -\frac{2}{3H}\right). \tag{45}$$

The metric (43) takes the following form

$$ds^2 = \sinh^4\left(\frac{3}{2}\,H\tau\right)\left(-c^2 d\tau'^2 + dz^2\right). \tag{46}$$

Or using (44)

$$ds^2 = \left[1 - \left(\frac{3}{2}\,H\tau'\right)^2\right]^{-2} \left(-c^2 d\tau'^2 + dz^2\right) \tag{47}$$

providing the conformal form of the FLRW metric in unified two dimensional spacetime.

On another hand, in four-dimensional spacetime, there is the parameter $\mathcal{H}$†

$$\mathcal{H}(\tau) = \frac{\dot{a}}{a} = \frac{v}{r} = \frac{\dot{V}}{4\pi r^3}\,. \tag{48}$$

Using (32) for $V(\tau)$ with the hyperbolic sine of half argument leads to

$$\mathcal{H}(\tau) = \frac{H}{\tanh\left(\frac{3}{2}\,H\tau\right)} \tag{49}$$

where the parameter belongs to the following interval

$$\mathcal{H}(\tau) \in (+\infty, H)\,. \tag{50}$$

Then the parameter can be written in terms of conformal time τ' as given by (44)

$$\mathcal{H}(\tau) = -\frac{3}{2}\,H^2\tau'\,. \tag{51}$$

This expression connects the "varying Hubble constant" with conformal time in unified 2D spacetime. The range of $\mathcal{H}(\tau)$ is from $+\infty$ to H and $\mathcal{H}(\tau)$ is the infinitely approaching value of H, as shown.

Interestingly that the metric (43) represents the embedding class two geometry, implying that the minimal number of dimensions of flat spacetime where it can be embedded is four. The reason why at least two additional dimensions are required is that the derivative $\dot{a}(\tau)$ has zero at $\tau = 0$, see [1, the Theorem 2.2].

9 The note on $3Hc$ and the number of spatial dimensions, the de Sitter metric

The appearance of the factor 3 in the value of the uniform acceleration (38) is closely related to the number of spatial dimensions. It can be demonstrated by the example of the de Sitter metric. Expressing the hyperbolic sine from the equation of motion (30) and substituting into the expression for proper velocity u^1 leads to

$$u(\eta) = c\,\frac{\eta}{\eta_0}\left(1 + \frac{2\eta_0}{\eta}\right)^{1/2}. \tag{52}$$

For far away distances when $\eta \gg \eta_0$ the second term in the equation can be neglected and using the value for η_0 from (31) it reduces to $u(\tau) = 3H\eta(\tau)$ with the solution

$$\eta(\tau) = a_1 \exp(3H\tau) \tag{53}$$

where a_1 can be set to the Rindler horizon distance η_0 as per (39). Then it becomes

$$V = \left(\frac{\beta}{3}\right) V_m \exp(3H\tau)\,. \tag{54}$$

*The metric clearly differs from the known form in comoving Rindler frame $ds^2 = -c^2\left(1+\alpha^2x^2\right)d\tau^2 + dx^2$ as the later uses different coordinate x that is defined locally in the observer's frame.

†Though the definition is the same as "varying Hubble constant" in the standard cosmology, their meanings have to be distinguished.

Using (8) and taking the cubic root result in

$$a(\tau) = \left(\frac{\beta}{3k}\right)^{1/3} \exp(H\tau) \tag{55}$$

which is the de Sitter metric where the factor 3 in the argument of the exponent disappears because of the cubic root. Interestingly pre-factor can not be unity in such way (the same can be shown by approximating (33)).

10 Coordinate time in 2D and in 4D spacetimes

Time is an arbitrary coordinate in gravitational theories including the GR [11] as it is not considered as absolute time. The model uses the proper time of the moving space τ that comes to the metric (2). The radial river velocity of the fluid / space v is the fluid's proper velocity in pseudo flat 4D Minkowski spacetime [3, 8] and v is the projection of proper velocity u_1 in 2D (t, η) as shown. However, the projection of coordinate velocity u_c in 2D (t, η) does not correspond to coordinate velocity of the fluid v_c in 4D because the Lorenz invariance in 2D cannot be applied to the Lorenz invariance is 4D. Therefore coordinate time in (t, η) is not synchronized with coordinate time in 4D (t', r, θ, ϕ). Such disagreement in coordinate times can be seen from the fact that time t in (t, η) implies how an observer residing at rest in $\eta = 0$ (so $r = 0$) measures its time. However, the coordinate time in 4D t' (that comes to the metric (1)) is time measured by static observer residing far away from the gravity $r = \infty$ (so $\eta = \infty$).

Whereas proper time τ of the comoving fluid in 2D is the same as proper time in 4D and such proper time invariance may imply invariance of the energy for coordinate transform from 2D to 4D but the topic requires further analysis. Coordinate time t' in four dimensional space time can be obtained from τ using the transform for the Gullstrand-Painlevé metric [3, 8]

$$d\tau = dt' - \frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1} dr \tag{56}$$

where τ is also proper time in 2D. As v represents proper velocity $(dr/d\tau)$ then dividing both sides by $d\tau$ it takes following form

$$dt' = \frac{d\tau}{1 - \frac{v^2}{c^2}} \tag{57}$$

Then the transform from proper time to coordinate time in 4D is given by respective integral using $v(\tau)$.

11 The dynamic of the Rindler flow along η

One dimensional flow with constant acceleration and velocity u provides certain simplification of the case study on the one hand. The analogue of one dimensional density for example becomes $\rho_\eta = m_0/\lambda_m$. However, some of the parameters like pressure can not be defined. The constant two-force acting on a fluid element is

$$F^i = m_0\,\alpha\left(\sinh\left(\frac{\alpha}{c}\tau\right), \cosh\left(\frac{\alpha}{c}\tau\right)\right) \tag{58}$$

where $i = 0, 1$ and $\alpha = 3Hc$ as per (38). Using definition for m_0 (25) the norm of the constant force is

$$|F| = \frac{9k}{2c}\,\hbar H^2\,. \tag{59}$$

It is easy to see that work done by such force at distance from 0 to the Rindler horizon given by (31) is exactly

$$|F|\,\eta_0 = m_0\,c^2 \tag{60}$$

and does not depend on values of β and k. This expresses the significance of the Rindler horizon distance in the frame of the model. The relativistic energy density for such fluid is $e = \rho_\eta c^2\gamma = \rho_\eta u^0 c$. The integration yields the total energy within the line segment $(0, \eta)$ as

$$E = \int_0^\eta e\,d\eta = \rho_\eta c \int_{\tau=0}^{\tau(\eta)} u^0 u^1 d\tau = \frac{m_0 c^4}{2\alpha\lambda_m}\cosh^2\left(\frac{\alpha}{c}\tau\right)\Big|_0^{\tau(\eta)}$$

$$= \frac{m_0 c^4}{2\alpha\lambda_m}\left(\cosh^2\left(\frac{\alpha}{c}\tau\right) - 1\right) \tag{61}$$

where in the last identity the value is taken at $\tau = 0$. Notable that the expression in brackets coincidences with $(u^1)^2$. Setting the hyperbolic cosine to 2 at distance η_0 as per (31) the total energy of the fluid from 0 to the Rindler horizon distance becomes

$$E(\eta_0) = \left(\frac{\beta}{2}\right) mc^2 \tag{62}$$

where $\alpha = 3Hc$ (38), (16) to express m and (31) were used. The energy invariance between 2D and 4D can be proposed based of the invariance for proper time τ between two spacetimes but it requires further analysis.

12 The continuity of the Rindler flow

The fluid flow with the relativistic uniform acceleration along η has many notable properties. As an example with the source placed at point $\eta = 0$ in case of incompressible fluid its strength is

$$\sigma = \frac{\partial m}{\partial t} = m_0\,\frac{\partial u}{\partial t} = 0\,. \tag{63}$$

However further along the coordinate such sink-source term is non-zero. It is easy to see using the equation of motion (30) for two points with initial distance λ_m (where we fix the initial line segment at $dt = \lambda_m/c$) then the distance between them increases with time as*

$$d\eta = \lambda_m \sinh\left(\frac{\alpha}{c}\tau\right). \tag{64}$$

In comoving frame of reference one can use proper velocity u^1 for the continuity equation. The divergence of proper velocity can be obtained as

$$\mathrm{div}(u^1) = \frac{\partial u^1}{\partial \eta} = \frac{\partial u^1}{\partial t}\,\frac{\partial t}{\partial \eta} = \frac{\alpha}{u_c} = \frac{\alpha}{c\tanh\left(\frac{\alpha}{c}\tau\right)}\,. \tag{65}$$

*Then the substitution of α from (38), using (17) leads to the element of the fluid growth in 3D as $V(\tau) = V_0 \sinh(3H\tau)$ which is exactly the same relation as suggested in [7] for the fluid parcel growth.

Lemma. The divergence of the proper velocity in 2D equals to divergence of the radial river velocity in 4D

$$\mathrm{div}(u^1) = \mathrm{div}(v)\,. \tag{66}$$

Proof. The radial velocity is irrotational as stated then

$$\mathrm{div}(v) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v\right) = \frac{2v}{r} + \frac{\partial v}{\partial r}\,. \tag{67}$$

Expressing v with u as given in (21)

$$\mathrm{div}(v) = \frac{V_m \beta H}{c}\,\frac{\partial u}{\partial r}\,\frac{1}{4\pi r^2} \tag{68}$$

where two identical terms dropped. As

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{\partial u}{\partial \eta}\,\frac{c}{V_m \beta H}\,4\pi r^2 \tag{69}$$

where (22) was used the substitution into (68) proves the lemma.

Combining (65), (66) and (49), using the value for α (26) and the trigonometric identities the divergence of the river velocity becomes

$$\mathrm{div}(v) = \frac{3\dot{a}}{2a}\left[1 + \left(\frac{a}{\dot{a}}H\right)^2\right]. \tag{70}$$

The equation provides the correspondence of the parameter $\mathcal{H}(\tau) = \dot{a}/a$ to the sink-source strength of fluid with constant density.

13 The limitations of the model

The first limitation of the model is that it does not provide any feasible solution for the Kerr-Newman neither for the Reissner–Nordström metrics. In the presented model, the rotation of the in 3D can not be distinguished in η coordinate because of the uncertainty of the volume V_0 represented as the spherical shell, as depicted in Fig. 1. Though it does not create any issue for the model because the Kerr-Newman river velocity does not have any dependency on angular coordinates (ϕ, θ) but only on radial coordinate as shown in [5]

$$v(r) = \left[\frac{2Gmr - Q^2}{r^2 + A^2}\right]^{1/2} \tag{71}$$

where A is the angular momentum per unit mass of a rotating mass, and Q is its charge. The model has difficulties in obtaining the analytic expressions in the same way for such velocity. There are two arguments to support the model, particularly is that the Kerr-Newman metric is a pure theoretical consequence of the GR and is not anyhow verified experimentally. The second argument is that the model is not unique in the sense that the coordinate η can be introduced differently but in the same manner for example

$$\gamma d\eta = dV\,\frac{\lambda_m}{V_0} \tag{72}$$

where γ is u^0 in the unified 2D spacetime. In such case spatial 3D coordinates (dV at right hand side) have "mixed" projection to both η and t (contrary to reviewed case where $\eta \to dV$ directly). Introduced in such way the river velocity for the SdS metric would be simply

$$v_p = v_c \gamma = \left(\frac{r_m^3 H^2}{r}\right)^{1/2}\left(1 + \frac{r^3}{r_m^3}\right)^{1/2} \tag{73}$$

where $k r_m^3 H^2 = 2Gm$. So the coordinate velocity is the Schwarzschild river velocity. Such alternative definition of η aligns coordinate time t in 2D and t' in 4D. The case for the mixed projection can be elaborated in future work.

14 Free fall velocity and symmetries

In the frame of the presented approach, the acceleration α along η has a positive value. Its projection to 4D results in positive radial velocity v in an outward direction (that in the Schwarzschild case corresponds to the negative deceleration in outward direction). The free-fall velocity v_{ff} is connected to the river velocity as $v_{ff} = -v$. The changing of sign in the acceleration α corresponds to the transform of the river velocity to free-fall velocity as $\alpha \to -\alpha \quad v \to v_{ff}$ Alternatively, the transform of the river velocity to free-fall velocity can be given via the change of sign of proper time τ because time reversal changes a sign of u and therefore it changes a sign of the radial river velocity v as per (20) $\tau \to -\tau \quad v \to v_{ff}$. However, such time reversal does not change a sign of the acceleration α. If one would extend the direction of η coordinate to the negative values (understanding that it would correspond to negative volume or negative ρ_0) then mirroring the coordinate η (to opposite direction) means the equivalently the change of sign of the acceleration as per the equation of motion (30) $\eta \to -\eta \quad \alpha \to -\alpha$.

15 Conclusions

The proposed analogy of unified two-dimensional spacetime brought a few convenient advantages to study the cosmological metrics and gravitation via the simplification. From the perspective of unified 2D spacetime the Schwarzschild gravity can be viewed as a non-relativistic approximation of flow with the constant acceleration. Then the relativistic considerations of such movement in unified 2D spacetime lead to the appearance of the repulsive Λ-term corresponding to the SdS metric. And this is far from being analogy as the case is only possible if the unified 2D spacetime is considered as *physical* spacetime. It can be interpreted as the "internal" spacetime of the moving fluid of the analog gravity and the River model.

As shown, the FLRW metric in unified 2D spacetime has the conformal form. The conformal time is connected to the parameter $\mathcal{H}(\tau)$ that is usually associated with the "varying Hubble constant". The parameter $\mathcal{H}$ varies from the infinity in the past to the Hubble constant, which will be approaching infinite time (49). Therefore the model has no place for the cosmological Big Crunch. The cosmological Big Bang is also absent. The model suggests that the Big Bang is going on continuously, equivalently signifying the emission of the fluid from the center of the point mass of every elementary particle where it is represented by the Rindler coordinate singularity at $\eta = 0$, $\tau = 0$. The Universe can be static as the equivalence of the metrics (1) and (2) is stressed.

The parallel of the model with the Conformal Quantum Mechanic that utilizes a 1D coordinate is yet to be analyzed. Possible outlook to the quantum properties of the Rindler fluid with constant force (59) (the linear potential) in unified 2D coordinates can be interesting. Embedding the electric charge to the metric in the frame of the model (where some of the parameters are to become imaginary) can be challenging.

Mathematical topics such as the topological coordinate transformation of 4D to 2D manifold and conformal mapping with the discrete maps in application to the presented model require further attention.

The exploration of additional coordinates is a strong trend since the foundation of Special Relativity. However, the opposite direction in the unification of known dimensions may also be surprisingly advantageous. The introduced unified 2D spacetime (t,η) via certain simplification offers a new perspective to look at gravitation and cosmology.

The presented intuitive approach reveals the significant parallel between gravity and motion in two-dimensional spacetime. As always, the analogy may be evidence of a hidden pattern in Nature; therefore, more thorough research and formal analysis are required.

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