Properties of Superdeformed Rotational Bands in the Perturbed SU(3) Limit of the sdg Interacting Boson Model

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The nuclear superdeformed bands in A∼ 190, A∼ 130 mass regions have been systematically analyzed by using the perturbed SU(3) limit of the interacting boson model. The g-bosons have been taken into consideration and the SU(3) symmetry is perturbed by introducing an interaction holding the SO(5) symmetry. A four parameters simple analytic formula for the eigenvalue equation has been derived. The spin determines of the studied superdeformed (SD) bands are considered from our previous works. The improved model parameters for each nucleus have been determined by operating a computer simulated search program so as to obtain a minimum root mean square divergence of the evaluating gamma ray transition energies and the observed ones. With these adopted model parameters the transition energies E_γ, the rotational frequencies $\hbar\omega$, the kinematic $J^{(1)}$ and dynamic $J^{(2)}$ moments of inertia have calculated and are in accordance with experimental data. The behavior of $J^{(1)}$ and $J^{(2)}$ as a function of $\hbar\omega$ have been studied. The calculated E_γ have been used to investigate the anomalous $\Delta I = 2$ staggering by considering the five point formula of Cederwall staggering parameter which represent the finite deviation calculation to the fourth order derivative of the transition energies at a determined spin.

1 Introduction

It was known that the interacting boson model (IBM) [1] with s and d bosons (sdIBM) is successful in studying the spectroscopic properties of low-lying collective states in heavy and medium nuclei. This simple sdIBM allows the utilization of the algebraic symmetries for approaching different type of nuclear spectra, known as dunamical symmetries U(5), SU(3) and O(6) which geometrically describe vibrational, axially deformed and gamma soft nuclei respectively. These three symmetry limits form a Casten triangle [2], that represent the nuclear phase diagram [3]. Transitions of shape phase between these vertices of Casten triangle were widely calculated along several isotopic chains [4–10]. Extended version of IBM where one includes the g-bosons in addition to s and d bosons to account for hexadecapole deformation of the nucleus is receiving a considerable attension of several research groups [11, 12]. This hexadecapole deformation is the second most important mulitipolarity in the description of nuclear properties in addition to the quadrupole deformation. An interest in this multipolarity is increased by the observation of the $\Delta I = 2$ energy staggering of superdeformed rotational bands (SDRB's) in some nuclei [13, 14], where nuclear spins with rotational sequences splitting by two may divide into two branches. Several theoretical attempts were made for the possible explanation of this $\Delta I = 2$ staggering phenomenon [15–25]. To describe the dynamical symmetries of nuclear states consisting of spdf bosons, it was found [26, 27] that one must begin with a supersymmetric group chain U(15,10) and ending at O(3) due to conservation of angular momentum passing through SU(3) limit of the sdg IBM which is a reasonable starting point to describe SD states in IBM [28]. The sdg IBM is well adopted for study of starting deformed and SD nuclei [15, 16, 26] there is seven different limits of SU(15) [29]. These limits can be splitted into two sets, the first set consists of the three limits which include only partial mixing between the bosons, however the second set consists of four limits which include a mixing of all bosons. If we consider the case of two s, d or g bosons, then the possible angular momenta are $L = 0^3, 2^4, 3, 4^4, 5, 6^2, 8$ where the exponent indicates the multiplicity. The $L = 3, 5$ states are pure dg configurations while the $L = 8$ states is pure g^2. All other states however are mixtures of s, d and g bosons. The difficulty with performing sdg IBM computations for normal deformed and superdeformed nuclei that have boson numbers $N = 12 - 16$ is that the core is too large, and the numerical methods (diagonalization) of the Hamiltonian is not possible. It was proved that the mathematical properties of the $SU(5)_{sdg}$ can be describe the deformed nuclei [30] because by using the intrinsic coherent states [11] the potential energy surface (PES) of the $SU(5)_{sdg}$ limit displays two minima. Since SDRB's are known in the second minimum of the potential well, this property was used [31] to justify an applications of $SU(5)_{sdg}$ limit in SD states. The group SU(3) which relates to the representations $[f_1, f_2, f_3]$ through $\lambda = f_1 - f_2$ and $\mu = f_1 - f_3$ is very important in studying the axial symmetric SDRB's. The one boson state belongs to the $(\lambda, \mu) = (4, 0)$ representation while the two bosons states belongs to (7,0), (4,2), (0,4) representation. To appear the $\Delta I = 2$ staggering, the SU(3) must be broken down by adding the $SO(5)_{sdg}$ symmetry as a perturbation. The aim of this work is to use this perturbed SU(3) of sdgIBM to investigate the main properties of superdeformed rotational bands in different nuclei and especially exhibit the $\Delta I = 2$ staggering in their transition energies.

2 Outline of the model

The states of SD bands can be classified in framework of supersymmetric group chain as:

$$\begin{array}{ccccccccccccc} U(m,n) & \supset & U_B(m) & \otimes & U_F(n) & \supset & ... & \supset & SO_{B+F}(3) & \otimes & SU_F(\grave{n}) & \supset & O(3) \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ [N] & & [N_B]_m & & [N_F]_n & & & & L & & S & & I \end{array}$$

The notation under those of groups are the corresponding irreducible (irrep) representation. The particles total number $N = N_F + N_B$ with N_F and N_B the fermion and boson numbers respectively. L is the effective core angular momentum and S is the total pseuduspin and I is the total spin of the nucleus. m is determined by the constituent of bosons, while n is determined by the single particle configuration of the fermions and $\grave{n}$ is the total pseudospin. Since the bosons to describe positive parity SD states should be s, d, g bosons [17, 20, 22] and p,f bosons are essential to show negative parity states [27], the space spanned by the single boson states is $\sum_{\ell}(2\ell + 1) = 1 + 3 + 5 + 7 + 9 = 25$ dimensions. So that, we have the group chain for the boson part

$$\begin{array}{ccccccccccccc} U_{sdgpf}(25) & \supset & U_{sdg}(15) & \otimes & U_{pf}(10) & \supset & SU_{sdg}(3) & \otimes & SU_{pf}(3) & \supset & SU(3) & \supset & O(3) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N_B] & & [N_{sdg}] & & [N_{pf}] & & (\lambda,\mu)_{sdg} & & (\lambda,\mu)_{pf} & & (\lambda,\mu) & & I \end{array}$$

The law-lying positive parity states are from the N_{sdg} bosons only, while negative parity states are one pf boson coupled states with $N_{sdg} = N - 1$ sdg bosons. There are also negative parity states formed by coupling odd number of pf bosons with residual sdg bosons and states of positive parity formed by even number of pf bosons with the sdg bosons. Here $N_B = N_{sdg} + N_{pf}$ with $N_{sdg} = 0, 1, 2, ..., N$ physically N is the number of positive parity bosons. All the irres can be determined with the branching rules [14] of the irres reduction. The reduction$SU(3)_{sdg} \otimes SU(3)_{pf} \supset SU(3)$ can be done in standard Young diagram method [10] and the reduction $SU(3) \supset O(3)$ is the Elliott rule [11]. We notice that for the positive parity states the results of the sdgIBM are still valid. The interaction Hamiltonian of the nucleus corresponding to the above chain takes the form

$$H = \epsilon C_1[U(15)] + kC_2[SU(3)] + cC_2[O(3)] \tag{1}$$

in which $C_k[G]$ is the k-order Casimir operator of the group G. The energy of the states can be formulated as

$$E(I) = E_0 + \epsilon N + k[\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu] + CI(I + 1) \tag{2}$$

the $C_2 [O(3)]$ operator gives the rotational structure. In varaible moment of inertia model [32], the moment of inertia is spin dependent, such that as I increases, the moment of inertia increase due to the antipairing effect. Therefore, Hamiltonian equation (1) can be written as

$$H = \epsilon C_1[U(15)] + kC_2[SU(3)] + C_0 \frac{C_2[O(3)]}{1 + f_1 C_2[O(3)] + f_2(C_2[O(3)])^2} \tag{3}$$

where the terms with f_1 and f_2 take into account many-body interactions which induce antipairing driving and pairing damping effects on the moment of inertia. The energy of the state I in a band considering only the relative excitation of the states in a rotational band is given by

$$E(I) = C_0 \frac{I(I + 1)}{1 + f_1[(I + 1)I] + f_2[(I + 1)I]^2} \tag{4}$$

To describe the superdeformed rotational bands, we break SU(3) symmetry by adding the symmetry $SO_{sdg}(5)$ as a perturbation to the Hamiltonian. Therefore, the excited energy of the state of positive parity with spin I in SD band is thus given by

$$E(I) = B[\tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1)] + \frac{C_0}{1 + f_1[(I + 1)I] + f_2[(I + 1)I]^2} I(I + 1) \tag{5}$$

The (τ_1, τ_2) is the irrep of SO(5) group. In practical τ_1, τ_2 being fixed with the branching rules of the irrep reduction as [21–24]

$$(\tau_1, \tau_2) = (\tfrac{I}{2}, 0) \quad \text{if} \quad I = 4k, 4k + 1 \quad (k = 0, 1, 2, ...)$$

$$(\tau_1, \tau_2) = (\tfrac{I}{2} - 1, 2) \quad \text{if} \quad I = 4k + 2, 4k + 3 \quad (k = 0, 1, 2, ..)$$

3 Analysis of ΔI = 2 staggering in transition energies in SD bands

In framework of collective model [33], the rotational frequency $\hbar\omega$, the kinematic moment of inertia ($J^{(1)}$) and the dynamic moment of inertia ($J^{(2)}$) calculated from γ-ray transition energies for SDRB's are given from the following definitions

$$\hbar\omega = \frac{1}{4}\left[E_\gamma(I + 2 \to I) + E_\gamma(I \to I - 2)\right] \qquad (MeV) \tag{6}$$

$$J^{(2)} = \frac{4}{E_\gamma(I + 2 \to I) - E_\gamma(I \to I - 2)} \qquad (\hbar^2 MeV^{-1}) \tag{7}$$

$$J^{(1)} = \frac{2I - 1}{E_\gamma(I \to I - 2)} \qquad (\hbar^2 MeV^{(-1)}) \tag{8}$$

Table 1: The adopted best model parameters C_0, B, f_1, f_2 obtained from the fitting procedure for the studied SD bands. The bandhead spin I_0 and the experimental lowest transition energy $E_\gamma(I_0 + 2 \to I_0)$ for each SD is also given.

SD band	I_0 ($\hbar$)	C_0 $\hbar^{-2}$ keV	B keV	f_1 $\hbar^{-2}$	f_2 $\hbar^{-4}$	E_γ (keV)
^{194}Tl(SD1)	14	0.503298E+01	0.18912E-02	0.326365E-03	-0.34134E-03	268.00
^{194}Tl(SD3)	12	0.522016E0+1	0.37473E-01	0.401374E-04	-0.39907E-08	240.50
^{194}Tl(SD5)	10	0.492810E+01	0.36833E-01	0.307779E-04	-0.42746E-08	187.90
^{130}Ce(SD2)	24	0.909181E+01	-0.34824E-02	0.171564E-04	-0.50224E-08	841.00
^{132}Ce(SD1)	30	0.647195E+01	-0.13947E-01	-0.299066E-04	0.34647E-10	808.55
^{132}Nd(SD1)	40	0.419310E+01	0.16107E-01	-0.547523E-04	-0.11468E-10	797.00
^{136}Sm(SD1)	30	0.640396E+01	0.51834E-03	-0.111011E-03	0.17709E-07	888.00

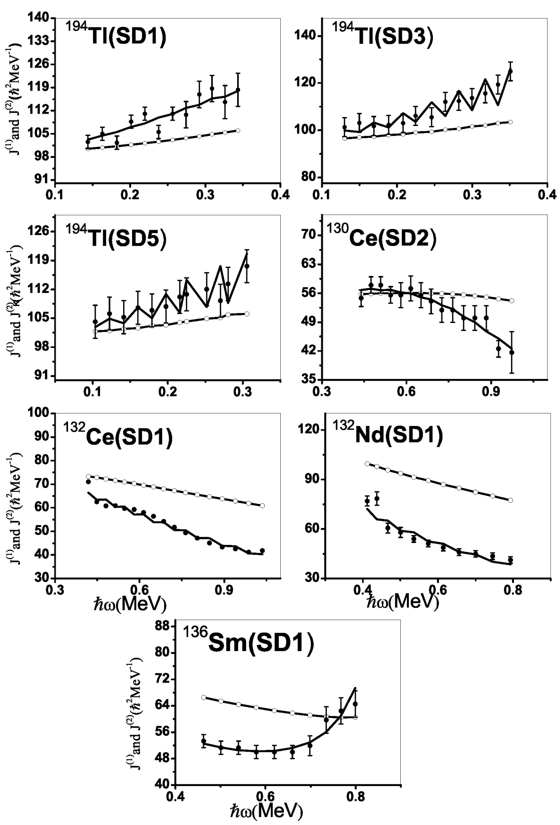

Fig. 1: The calculated results of the kinematic $J^{(1)}$ (open circles) and dynamic $J^{(2)}$ (solid curves) moments of inertia plotted as a function of rotational frequency $\hbar\omega$ for the studied SD bands and the comparison with experimental data for $J^{(2)}$ (closed circles with error bars)

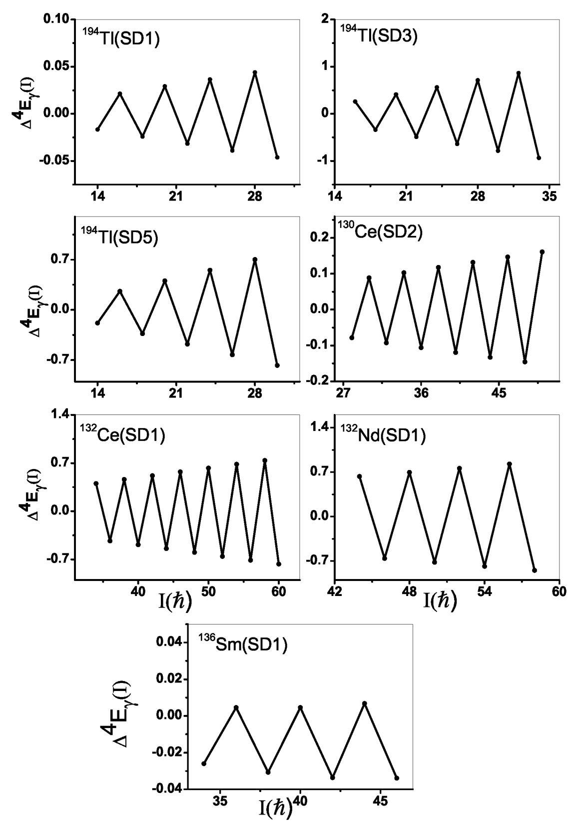

Fig. 2: The calculated $\Delta I = 2$ staggering quantity $\Delta^4 E_\gamma$ obtained by the five point formula as a function of spin for the studied SD bands.

The anomalous $\Delta I = 2$ staggering phenomenon was found in several SD bands [17,18]. Sequences of states which are differing by four units of angular momentum displace relative to each other was shown in superdeformed rotational bands. That is, the SD band can be seen as two sequences of cases with values of spin $I + 4n$ and $I + 4n + 2 (n = 1, 2, 3, ...)$, respectively. This is commonly called $\Delta I = 4$ bifurcation, because the bands divide into two branches with levels differing in spin by $4\hbar$. To explore this $\Delta I = 2$ staggering, the deviation of the $\gamma-$ray energies from a smooth reference $\Delta^4 E_\gamma(I)$ was determined by Cederwall [12], by calculating the finite difference approximation of the fourth order derivation of the $\gamma-$ray energies E_γ at a given spin I by

$$\Delta^4 E_\gamma^{ref}(I) = \frac{1}{16}\Big[E_\gamma(I-4) - 4E_\gamma(I-2) + 6E_\gamma(I) - 4E_\gamma(I+2) + E_\gamma(I+4)\Big] \tag{9}$$

with $E_\gamma(I) = E_\gamma(I) - E_\gamma(I-2)$. The formula (9) contains five energies of consecutive transition and is denoted by the five point formula.

4 Numerical calculations and discussion

For each band of our studied SDRB's, the spin of the bandhead I_0 is taken from our previous works [19–25]. The model parameters C_0, B, f_1, f_2 are determined by using a computer simulated search program in order to obtain a minimum root-mean square (rms) deviation of the calculated transition energies $E_\gamma^{cal}(I)$ from the experimental one $E_\gamma^{exp}(I)$, we employed the common definition of χ

$$\chi = \frac{1}{N}\sqrt{\sum_{i=1}^{N}\left|\frac{E_\gamma^{exp}(I_i) - E_\gamma^{cal}(I_i)}{\delta E_\gamma^{exp}(I_i)}\right|^2} \tag{10}$$

where N is the number of the data points entering into the fitting procedure and $\delta E_\gamma^{exp}(I_i)$ are the experimental errors in $\gamma-$ray energies. Table(1) shows the predicted bandhead spins and the best values of the model parameters C_0, B, f_1, f_2 for each band. Also indicated in Table(1) are the lowest $\gamma-$ray transition energies $E_\gamma(I + 2 \to I_0)$. Using the adopted model parameters, the transition energies E_γ, rotational frequencies $\hbar\omega$, the kinematic $J^{(1)}$ and dynamic $J^{(2)}$ moments of inertia of our selected SD bands are obtained. A very good agreement between the calculated and the experimental values is obtained which gives good support to the model. The kinematic $J^{(1)}$ and dynamic $J^{(2)}$ moments of inertia are plotted as a function of rotational frequency $\hbar\omega$ in Figure(1) compared to the experimental ones. In A∼190 mass region, $J^{(1)}$ values are found to be smaller than $J^{(2)}$ and $J^{(2)}$ exhibits a gradual increases with increasing $\hbar\omega$, while in A∼130 the values of $J^{(2)}$ are smaller than that the corresponding values of $J^{(1)}$ for all ranges of frequencies and $J^{(2)}$ mostly decrease with a great deal of variation from nucleus to nucleus. Another result in the present work is the observation of a $\Delta I = 2$ staggering effect in $\gamma-$ray energies $E_\gamma(I + 2 \to I)$ in the studied SDRB's. The the staggering pattern is illustrated in Figure(2) where the staggering parameters $\Delta^4 E_\gamma(I)$ introduced by Cederwall et al [14] defined as the fourth derivative of E_γ are presented as a function of rotational frequency $\hbar\omega$. A significant zigzag has been observed the resulting numerical values for each band are listed in Tables(2 and 3).

5 Conclusion

The SDRB's namely ^{194}Tl(SD1, SD3, SD5), ^{130}Ce(SD1), ^{132}Nd and ^{136}Sm(SD1) are studied in the version of the perturbed SU(3) limit of sdgIBM with supersymmetry scheme including many body interaction. The bandhead spins are taken from our previous works while the model parameters are adjusted by fitting procedure in order to minimize the relative root mean square deviation between experimental transition energies E_γ^{exp} and the calculated ones E_γ^{cal}. Excellent agreement are given which gives good support to the proposed model. Rotational frequencies, kinematic $J^{(1)}$ and dynamic $J^{(2)}$ moments of inertia are calculated and the evolution of $J^{(1)}$ and $J^{(2)}$ with $\hbar\omega$ are studied. The calculated E_γ are used to investigate the occurrence of a $\Delta I = 2$ staggering effect in the studied SDRB's by using the fourth order derivative of the $\gamma-$ray transition energies. A large amplitude staggering pattern is found in all the studied SDRB's.

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Table 2: Calculated γ−ray transition energy E_γ(keV) for the studied SD bands and the comparison with experiment and the calculated results of the $\Delta I = 2$ energy staggering parameter $\Delta^4 E_\gamma(I)$ obtained by five point formula.

^{194}Tl(SD1)				^{194}Tl(SD3)			
$I(\hbar)$	$E_\gamma(exp)$	$E_\gamma(cal)$	$\Delta^4 E_\gamma(I)$	$I(\hbar)$	$E_\gamma(exp)$	$E_\gamma(cal)$	$\Delta^4 E_\gamma(I)$
14	268.0	268.6751		12	240.5	238.2344	
16	307.0	307.4222		14	280.0	278.2792	
18	345.1	345.7051	-0.0165531	16	318.8	318.5983	0.262866
20	384.2	383.6179	0.0213944	18	358.1	357.3467	-0.33660
22	421.0	420.9822	-0.0241556	20	39.2	396.8819	0.412796
24	457.0	457.9697	0.0290825	22	425.3	434.1794	-0.486442
26	494.9	494.3612	-0.0316294	24	473.0	472.8180	0.562820
28	530.9	530.4057	0.0365431	26	510.9	508.5947	-0.636306
30	567.0	565.8415	-0.0389094	28	546.6	546.3097	0.712723
32	601.2	600.9968	0.0438844	30	582.2	580.5822	-0.786148
34	634.9	635.5733	-0.0463116	32	617.4	617.4370	0.862580
36	669.8	669.9782		34	652.0	650.3192	-0.935853
38	703.6	703.8759		36	685.5	686.4767	
				38	717.5	718.1790	

^{194}Tl(SD5)				^{130}Ce(SD2)			
$I(\hbar)$	$E_\gamma(exp)$	$E_\gamma(cal)$	$\Delta^4 E_\gamma(I)$	$I(\hbar)$	$E_\gamma(exp)$	$E_\gamma(cal)$	$\Delta^4 E_\gamma(I)$
10	187.9	186.6115		24	841.0	842.347	
12	226.3	225.4885		26	914.0	912.752	
14	264.0	263.6266	-0.183757	28	983.0	982.836	-0.0785506
16	302.0	302.1677	0.258395	30	1052	1053.335	0.0890181
18	339.2	339.3166	-0.330951	32	1124	1123.728	-0.0921319
20	376.6	377.4083	0.405704	34	1196	1194.918	0.10318
22	413.7	413.4865	-0.478076	36	1266	1266.335	-0.105713
24	450.0	451.0842	0.553079	38	1338	1339.059	0.117491
26	486.1	486.0854	-0.625427	40	1412	1412.478	-0.119078
28	521.8	523.2236	0.700607	42	1489	1487.861	0.132018
30	558.4	557.2253	-0.772767	44	1566	1564.571	-0.132682
32	593.7	594.0268		46	1646	1644.083	0.146972
34	627.7	627.2002		48	1726	1725.751	-0.145203
				50	1806	1811.279	0.161256
				52	1900	1900.046	
				54	1996	1994.015	

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Table 3: Same as Table 2 but for ^{132}Ce(SD1), ^{132}Nd(SD1) and ^{136}Sm(SD1).

^{132}Ce(SD1)				^{132}Nd(SD1)			
$I(\hbar)$	$E_\gamma(exp)$	$E_\gamma(cal)$	$\Delta^4 E_\gamma(I)$	$I(\hbar)$	$E_\gamma(exp)$	$E_\gamma(cal)$	$\Delta^4 E_\gamma(I)$
30	808.55	804.9409		40	797.0	793.4394	
32	864.85	865.2221		42	849.0	848.8271	
34	928.80	928.3618	0.404878	44	900.0	909.5820	0.630371
36	994.63	991.3491	-0.431978	46	966.0	971.1123	-0.658327
38	1060.32	1057.651	0.460603	48	1035.0	1038.912	0.69562
40	1127.27	1123.824	-0.487641	50	1109.0	1107.942	-0.721866
42	1194.72	1193.792	0.516664	52	1187.0	1184.293	0.760989
44	1263.63	1263.680	-0.543336	54	1269.0	1262.507	-0.785341
46	1334.56	1337.876	0.572326	56	1356.0	1349.300	0.826478
48	1408.34	1412.078	-0.599181	58	1445.0	1438.824	-0.847901
50	1452.67	1491.137	0.629027	60	1537.0	1538.453	
52	1566.70	1570.322	-0.655639	62	1634.0	1641.996	
54	1651.49	1654.962	0.685363				
56	1740.29	1739.900	-0.71106				
58	1832.64	1830.941	0.740601				
60	1926.50	1922.515	-0.766111				
62	2023.50	2020.902					
64	2119.00	2120.123					

^{136}Sm(SD1)			
$I(\hbar)$	$E_\gamma(exp)$	$E_\gamma(cal)$	$\Delta^4 E_\gamma(I)$
30	888	887.0356	
32	963	963.2182	
34	1041	1041.0786	-0.0260325
36	1119	1120.2441	0.0045813
38	1199	1199.9257	-0.0306712
40	1279	1279.4072	0.00451688
42	1359	1357.4814	-0.0336294
44	1436	1433.0136	0.00677437
46	1503	1504.3310	-0.0338756
48	1567	1569.8691	
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