A Note on the Barut Second-Order Equation

Valeriy V. Dvoeglazov

UAF, Universidad Autónoma de Zacatecas, Apartado Postal 636, Zacatecas 98061 Zac., México
E-mail: valeri@fisica.uaz.edu.mx

The second-order equation in the $(1/2, 0)\oplus(0, 1/2)$ representation of the Lorentz group has been proposed by A. Barut in the 70s [1]. It permits to explain the mass splitting of leptons $(e, \mu, \tau)$. The interest is growing in this model (see, for instance, the papers by S. Kruglov [2] and J. P. Vigier et al. [3, 4]). We note some additional points of this model.

The Barut main equation is

$$[i\gamma^\mu\partial_\mu + \alpha_2\sigma^\mu\partial_\mu - \kappa] \Psi = 0 \tag{1}$$

where $\alpha_2$ and $\kappa$ are the constants later related to the anomalous magnetic moment and mass, respectively. The matrices $\gamma^\mu$ are defined by the anticommutation relation:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{2}$$

$g^{\mu\nu}$ is the metrics of the Minkowski space, $\mu, \nu = 0, 1, 2, 3$. The equation represents a theory with the conserved current that is linear in 15 generators of the 4-dimensional representation of the $O(4, 2)$ group, $N_{ab} = \frac{1}{2}\gamma_a\gamma_b, \gamma_a = \{\gamma_\mu, \gamma_5\}$. Instead of 4 solutions, (1) has 8 solutions with the correct relativistic relation $E = \pm \sqrt{\mathbf{p}^2 + m^2}$. In fact, it describes states of different masses (the second one is $m_2 = 1/\alpha_2 - m_1 = m_s(1 + 3/2\alpha), \alpha$ is the fine structure constant), provided that the certain physical condition is imposed on $\alpha_2 = (1/m_1)(2\alpha/3)(1 + 4\alpha/3)$, the parameter (the anomalous magnetic moment should be equal to $4\alpha/3$). One can also generalize the formalism to include the third state, the $\tau$-lepton [1b]. Barut has indicated the possibility of including $\gamma_5$ terms (e.g. $\sim \gamma_5\kappa$).

The most general form of spinor relations in the $(1/2, 0)\oplus(0, 1/2)$ representation has been given by Dvoeglazov [5]. It was possible to derive the Barut equation from first principles [6]. Let us reveal the connections with other models. For instance, in [3, 7] the following equation has been studied:

$$\left[(i\partial - e\hat{A})(i\partial - e\hat{A}) - m^2\right] \Psi = 0 \tag{3}$$

for the 4-component spinor $\Psi = \psi A; \psi$ is the 4-vector potential; $e$ is electric charge; $eA\mu$ is the electromagnetic tensor. $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]...$ This is the Feynman-Gell-Mann equation. In the free case we have the Lagrangian (see Eq. (9) of [3c]):

$$\mathcal{L}_0 = (i\partial^\mu\bar{\Psi})(i\partial^\mu\Psi) - m^2\Psi\bar{\Psi} \tag{4}$$

Let us re-write (1) into the form:

$$[i\gamma^\mu\partial_\mu + a\sigma^\mu\partial_\mu + b] \Psi = 0 \tag{5}$$

So, one should calculate $(p^2 = p_0^2 - \mathbf{p}^2)$

$$\text{Det} \left( \begin{array}{cc} b - a\sigma \cdot \mathbf{p} & p_0 + \sigma \cdot \mathbf{p} \\ p_0 - \sigma \cdot \mathbf{p} & b - a\sigma \cdot \mathbf{p} \end{array} \right) = 0 \tag{6}$$

in order to find energy-momentum-mass relations. Thus, $[(b - a\sigma^2)^2 - p^2]^2 = 0$ and if $a = 0, b = \pm m$ we come to the well-known relation $p^2 = p_0^2 - \mathbf{p}^2 = m^2$ with four Dirac solutions. However, in the general case $a \neq 0$ we have

$$p^2 = \frac{(2ab + 1) \pm \sqrt{4ab + 1}}{2a^2} > 0 \tag{7}$$

that signifies that we do not have tachyons. However, the above result implies that we cannot just put $a = 0$ in the solutions, while it was formally possible in (5). When $a \to 0$ then $p^2 \to \infty$; when $a \to \pm \infty$ then $p^2 \to 0$. It should be stressed that the limit in the equation does not always coincide with the limit in the solutions. So, the questions arise when we consider limits, such as Dirac $\to$ Weyl, and Proca $\to$ Maxwell. The similar method has also been presented by S. Kruglov for bosons [8]. Other fact should be mentioned: when $4ab = -1$ we have only the solutions with $p^2 = 4b^2$. For instance, $b = m/2, a = -1/2m$, $p^2 = m^2$. Next, I just want to mention one Barut omission. While we can write

$$\frac{\sqrt{4ab + 1}}{a^2} = m_2^2 - m_1^2, \text{ and } \frac{2ab + 1}{a^2} = m_2^2 + m_1^2 \tag{8}$$

but $m_2$ and $m_1$ should not necessarily be associated with $m_{\mu/e}$ (or $m_{\tau}$). They may be associated with their superpositions, and applied to neutrino mixing, or quark mixing.

The lepton mass splitting has also been studied by Markov [9] on using the concept of both positive and negative masses in the Dirac equation. Next, obviously we can calculate anomalous magnetic moments in this scheme (on using, for instance, methods of [10, 11]).

We previously noted:

- The Barut equation is a sum of the Dirac equation and the Feynman-Gell-Mann equation.
- Recently, it was suggested to associate an analogue of (4) with dark matter, provided that $\Psi$ is composed of $\tilde{a}$ has dimensionality [1/m], $b$ has dimensionality [m].
the self/anti-self charge conjugate spinors, and it has the dimension [energy]$^{1}$ in the unit system $c = \hbar = 1$. The interaction Lagrangian is $\mathcal{L}^H \sim g \overline{\Psi} \Psi \phi^2$, $\phi$ is a scalar field.

- The term $\sim \overline{\Psi} \sigma^{\mu\nu} \Psi F_{\mu\nu}$ will affect the photon propagation, and non-local terms will appear in higher orders.
- However, it was shown in [3b,c] that a) the Mott cross-section formula (which represents the Coulomb scattering up to the order $\sim e^2$) is still valid; b) the hydrogen spectrum is not much disturbed; if the electromagnetic field is weak the corrections are small.
- The solutions are the eigenstates of the $\gamma^5$ operator.
- In general, the current $J_0$ is not the positive-defined quantity, since the general solution $\Psi = c_1 \Psi_+ + c_2 \Psi_-$, where $[i\gamma^\mu \partial_\mu \pm m] \Psi_\pm = 0$, see also [9].
- We obtained the Barut-like equations of the 2nd order and 3rd order in derivatives.
- We obtained dynamical invariants for the free Barut field on the classical and quantum level.
- We found relations with other models (such as the Feynman-Gell-Mann equation).
- As a result of analysis of dynamical invariants, we can state that at the free level, the term $\sim \partial_\mu \overline{\Psi} \sigma_{\mu\nu} \partial_\nu \Psi$ in the Lagrangian does not contribute.
- However, the interaction terms $\sim \overline{\Psi} \sigma_{\mu\nu} \partial_\nu \Psi A_\mu$ will contribute when we construct the Feynman diagrams and the $S$-matrix. In the curved space (the 4-momentum Lobachevsky space), the influence of such terms has been investigated in the Skachkov work [10,11]. Briefly, the contribution will be such as if the 4-potential were to interact with some “renormalized” spin. Perhaps, this explains why Barut used the classical anomalous magnetic moment $g \sim 4\alpha/3$ instead of $\alpha/2\pi$.

Acknowledgements

The author acknowledges discussions with participants of recent conferences.

Received on November 17, 2020

References