

A Survey on Uncertainty Relations and Quantum Measurements: Arguments for Lucrative Parsimony in Approaches of Matters

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This survey tries to investigate the truths and deficiencies of prevalent philosophy about Uncertainty Relations (UR) and Quantum Measurements (QMS). The respective philosophy, known as being eclipsed by unfinished controversies, is revealed to be grounded on six basic precepts. But one finds that all the respective precepts are discredited by insurmountable deficiencies. So, in regard to UR, the alluded philosophy discloses oneself to be an unjustified mythology. Then UR appear either as short-lived historical conventions or as simple and limited mathematical formulas, without any essential significance for physics. Such a finding reinforces the Dirac's prediction that UR "*in their present form will not survive in the physics of future*". The noted facets of UR motivate reconsiderations of associated debates on QMS. Mainly one reveals that, properly, UR have not any essential connection with genuine descriptions of QMS. For such descriptions, it is necessary that, mathematically, the quantum observables to be considered as random variables. The measuring scenarios with a single sampling, such are wave function collapse or Schrödinger's cat thought experiment, are revealed as being useless inventions. We propose to describe QMS as transmission processes for stochastic data. Note that, for existing quantum debates, the above UR–QMS revaluations, offer a few arguments for lucrative parsimony in approaches of matters. The unlucrative aspects of those debates have to be reconsidered too, probably in more or less speculative visions.

Motto 1: "I think one can make a safe guess that uncertainty relations in their present form will not survive in the physics of future."

P. A. M. Dirac, 1963

Motto 2: "The word 'measurement' has had such a damaging effect on the discussions that ... it should be banned altogether in quantum mechanics."

J. S. Bell, 1990

Foreword

A. The present review-study germinates from some of our preceding more modest investigations some of them already published in this journal, Progress in Physics. Also, it was influenced by a number of opinions published by other scientists (opinions which, usually, are ignored in mainstream literature).

In the main, the study was stimulated by the known existence of numerous debates (unfinished controversies on unelucidated questions) regarding the foundations and interpretation of Quantum Mechanics (QM). The considered debates refer mainly to the significance of Uncertainty Relations (UR) and to the associated descriptions of Quantum Measurements (QMS). By their obstinate persistence, the mentioned debates delay and obstruct the desired (and expected) clarifications about some basic aspects of QM.

Within the here emerged text, we try to gather, systematize, improve, consolidate and mainly to present more

argued our non-conventional viewpoints about the existing prevalent debates on UR, QMS and QM.

B. The here proposed article approaches step-by-step the following main items:

- i.1 A consistent Introduction which points out:
 - (a) The nowadays existence of unfinished debates (disputes and controversies) about the meaning of UR and description of QMS;
 - (b) The today necessity for search the truth about own philosophy of UR and description of QMS, regarded as relevant pieces for foundations/interpretation of QM;
- i.2 An inventory section which identifies the Basic Precepts of the prevalent philosophy regarding UR and QMS;
- i.3 A large section about most important deficiencies of the mentioned precepts. Within the respective section we concern on:
 - (a) Detailed examinations of deficiencies specific to the respective precepts;
 - (b) Elucidation, piece by piece, of the real value/meaning for each of the pointed out deficiencies;
- i.4 A first concluding section about the true significance of UR. In that section the current prevalent interpretation of UR is proved to be nothing but a veritable myth without any special or extraordinary value for physics. But such a proof reinforces the Dirac's prediction that UR

“in their present form will not survive in the physics of future”;

- i.5 A section containing considerations on description of QMS. The respective considerations are done in the light of the debates about deficiencies of dominant philosophy about UR. Also the measuring scenarios with a single sampling, such are wave function collapse or Schrödinger’s cat thought experiment, are revealed as being superfluous fictions. We argue that the QMS descriptions should be approached additionally comparatively with the description and interpretation of UR. They must be discussed in new insights by regarding the measurements as transmission processes for stochastic data. (see our examples from Subsections 5.2 and 5.4 or from Appendices F and G);
- i.6 A final section with some concluding remarks;
- i.7 A supplementary section of Appendices containing:
- Technical/computational details — in seven cases, respectively;
 - A copy of “A private letter from the late scientist J. S. Bell to the author”.

C. Notes:

- I. Through the elucidations referred to in item i.3 we offer genuine solutions for some controversial theoretical problems such are:
- The adequate form of UR for the supposed rebellious pairs of observables: $L_z-\varphi$ (angular momentum — azimuthal angle), $N-\phi$ (number-phase) and $E-t$ (energy-time);
 - The case of macroscopic operators;
 - The uniqueness (individuality) of Planck’s constant;
- II. In its essence, the suggested revaluation of UR and QMS philosophy does not disturb in any way the basic lucrative framework of usual QM (which keeps its known specific concepts, principles, theoretical models, computing rules and studied systems); Moreover, I try to give arguments for lucrative parsimony in approaches of QM matters; I believe that, to some extent, such a revaluation of UR-QMS prevalent philosophy can be beneficent for interpretation and understanding of QM. Potentially that revaluation can bring at least a modest contribution to non-conventional investigations of some open questions regarding views about UR, QMS, and QM.
- III. My article tries to clarify certain past misunderstandings, of historical, philosophical, and cultural essence, which still persists in activities (of publishing and mainly of teaching nature), connected with QM;
- IV. As a significant aspect, in my paper, the discussions are presented and detailed in forms accessible to readers with knowledge of QM at a not-advanced level. That is why in the version proposed here the article was con-

ceived (especially through a number of detailed Appendices) as an accessible teaching material for those interested in QM education at undergraduate/graduate levels.

D. I think that, by its theme, style and writing level, my paper ensures the following desiderata:

- It approaches representative methodological and philosophical topics concerning the structure and the growth (interpretation and foundations) of QM investigated as a significant constituent of natural sciences;
- It can give a starting forum for the exchange of views and ideas among readers interested, in foundations of QM regarded as an important constituent of modern sciences;
- It identifies and highlights foundational issues, suggesting constructive and genuine solutions for approached problems;
- It offers a number of original opinions concerning some controversial theoretical/philosophical scientific problems;
- It initiates and develops discussions on the philosophy and epistemology of physics, at a level accessible to a wide class of readers (scientists, teachers and even students in physics, mathematics, chemistry or philosophy);
- It provides an argued appeal toward an increasing research field, namely to the one regarding the non-conventional approach of QM interpretation and foundations.

Given the above-mentioned aspects, I think that my article can offer a modest contribution to newly rising investigations on non-conventional views in quantum physics.

Braşov, November 26, 2020

Spiridon Dumitru

1 Introduction

Nearly a century until nowadays, in the publications regarding Quantum Mechanics (QM) and even other areas, have persisted discussions (debates and controversies) about the meaning of Uncertainty Relations (UR). Moreover UR in their entirety were ranked to a status of fundamental concept named Uncertainty Principle (UP) (for a bibliography of the better known specific publications see [1–12]). Mostly the respective discussions have credited UR/UP with considerable popularity and crucial importance, both in physics and in other domains. The mentioned importance was highlighted by compliments such as:

- UR are “*expression of the most important principle of the twentieth century Physics*” [13],
- UP is “*one of the cornerstones of quantum theory*” [9];

- UP “*epitomizes quantum physics, even in the eyes of the scientifically informed public*” [7].

But, as a fact, in spite of such compliments, in scientific literature of our days the essential aspects regarding UR/UP remain as unsolved and misleading questions. Today keeps their topicality many critiques reported during last decades, like the next ones:

- UR “*are probably the most controverted formula in the whole of the theoretical physics*” [14];
- “*Still now, 80 years after its inception, there is no general consensus over the scope and validity of this principle (‘UP’)*” [7];
- “*Overcoming the early misunderstanding and confusion, the concept (notion of uncertainty — i.e. of UR/UP) “grew continuously and still remains an active and fertile research field”* [8].

Note that the above reminded appreciations (compliments and critiques) regard mainly the own essence (intrinsic meaning) of UR/UP. But, within many texts about QM fundamentals, one finds also an adjacent topic which, historically, is a direct sub-sequence of the debates about the mentioned essence. The respective topic refers to the significance and description of Quantum Measurements (QMS).

Marked by the previously noted points, during the decades, the discussions about UR and QMS meaning and implications have generated a true prevalent philosophy (i.e. “*a group of theories and ideas related to the understanding of a particular subject*” [15]). For almost a century, the respective philosophy dominates in mainstream physics publications and thinking. It obstructs (delays) the expected progresses in clarifying some of main aspects regarding the fundamentals/interpretation of QM respectively the essentials of QMS problem. Add here the more alarming observation [16] that: “*there is still no consensus on ... interpretation and limitations of QM*”. Then it becomes of immediate interest to continue searches for finding the truth about own essence and consecutive topics of the UR/UP and QMS matters.

A search of the alluded type can be done (or facilitated at least) by a pertinent survey on deficiencies of the mentioned philosophy. Such a survey (of modest extent) we intend to present in this article. Our survey tries firstly to identify the basic elements of nowadays prevalent views within UR and QMS philosophy. Afterward we will investigate truth and value of the respective elements. Within the investigation we promote a number of re-considerations regarding the conventional (and now dominant) views about UR and QMS matters. Mainly we reveal the fact that the alluded views are discredited (and denied) by a whole class of insurmountable deficiencies, overlooked in the mainstream literature. So our survey aims to represent an unconventional analysis of the actual dominant philosophy about UR and QMS. The above-announced analysis germinated from some of our preceding

investigations (see [17–21] and references). **Also, it was stimulated by a number of opinions due to other scientists (usually the respective opinions are ignored in dominant literature, but here they are highlighted by specifying the proper bibliographic sources).** Through the present survey, we try to gather, extend, systematize, improve and consolidate the results of our mentioned investigations in order to present a more argued viewpoints about the approached topics.

In our survey, when it is usefully, we will appeal to the so called ‘parsimony principle’ (or ‘law ’). The respective principle (known also as Ockham’s razor) will be applied as a heuristic method of simplicity which can be summarized [22] by the next two desiderata:

- “*Of two competing theories, the simpler explanation of an entity is to be preferred*”.
- “*Entities are not to be multiplied beyond necessity*”.

The mentioned principle will be accounted for in order that the text to be easy understood for readers (including students) not highly specialized.

By the present article-survey, through adequate arguments and details, we try to elucidate what is in fact the true meaning of UR, respectively to evaluate the genuine scientific aspects regarding QMS.

From the conclusions resulting from this survey the most important one is that, in its entirety, the actual prevalent philosophy about UR must be regarded as a veritable myth without any special or extraordinary status/significance for physics. This because, in reality, the UR reveal themselves to be nothing but short-lived historical conventions (in empirical, thought-experimental version) or simple and restricted formulas (in theoretical approach). But such a conclusion come in consonance, from another perspective, with the Dirac’s guess [23] that: “*uncertainty relations in their present form will not survive in the physics of future*”.

Add here the fact that, essentially, the above mentioned re-evaluation of UR and QMS philosophy does not disturb in any way the basic framework (principles, concepts, models and working rules) of usual QM. Furthermore, the QMS description remains as a distinct and additional subject comparatively with the elements of QM in itself. Add here the observation that, for existing quantum debates, the above UR–QMS revaluations give a few arguments for lucrative parsimony in approaches of matters. The unlucrative aspects of those debates have to be reconsidered, probably in more or less speculative visions.

The mentioned description of QMS requires to regard quantum observables* as true random variables. Also it must

***Drafting specifications:** (i) In the next parts of this article, for naming a physical quantity, we shall use the term “*observable*” (promoted by the UR and QMS philosophy literature), (ii) Also, according to the mainstream publications, we adopt the titles “*commuting*” or “*non-commuting*” observables for the QM quantities described by operators which “*commute*” respec-

be dissociated of some fictive QMS scenarios with a unique sampling (such scenarios are schema with wave function collapse and Schrödinger's cat thought experiment). We recommend to describe QMS as transmission processes of stochastic data.

2 Basic precepts of UR–QMS prevalent philosophy

Firstly it must be pointed out the fact that, in spite of its prevalence inside of nowadays scientific debates, the actually dominant philosophy about UR and QMS germinates mainly from an old doctrine which can be called Conventional Interpretation of UR (CIUR). The mentioned doctrine (or dogma) was initiated by the Copenhagen School founders and, subsequently, during nine decades, it was promoted (or even extrapolated) by the direct as well indirect partisans (conformists) of the respective school. Currently CIUR enjoys of a considerable acceptance, primarily in QM studies but also in other thinking areas. Moreover, today, within the normative (mainstream/authoritarian) physics publications, CIUR dominates the leading debates about foundations and interpretation of QM.

But as a notable fact, in publications, CIUR doctrine, as well as most aspects of UR and QMS philosophy, are presented rather through independent or disparate assertions but not through a complete and systematized set of clearly defined “precepts” (considered as “beliefs ... accepted as authoritative by some group or school” [24]). That is why, for a fruitful survey of the UR–QMS philosophy, it is of direct interest to identify such an set of **Basic Precepts (BP)** from which the mentioned assertions turn out to be derived or extrapolated. Note that the aforesaid set of precepts (i.e. the true core of CIUR doctrine along with prevalent philosophy of UR and QMS) can be collected by means of a careful examination of the today known publications. In its essence the respective collection can be presented as follows.

The history regarding Conventional Interpretation of UR (CIUR) began with two main generative elements which were the following ones:

(i) Heisenberg's “Thought-Experimental” (TE) relation:

$$\Delta_{TEA} \cdot \Delta_{TEB} \cong \hbar \quad \text{or} \quad \Delta_{TEA} \cdot \Delta_{TEB} \geq \hbar; \quad (1)$$

(ii) Robertson-Schrödinger relation of theoretical origin:

$$\Delta_{\Psi A} \cdot \Delta_{\Psi B} \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle_{\Psi} \right|. \quad (2)$$

For introducing relation (1) in [25, 26] were imagined some “Thought Experiments” (TE) (or “gedanken” experiments). The respective TE referred on simultaneous measurements of two (canonically) conjugate observables A and B regarding a same quantum micro-particle. As such pairs of two

tively “do not commute”, (iii) For improving fluency of our text some of the corresponding mathematical notations, formulas and proofs are summarized briefly and unitary in few Appendices located in the final of the article.

observables were considered coordinate q and momentum p respectively time t and energy E . Then the quantities Δ_{TEA} and Δ_{TEB} were indicated as corresponding “uncertainties” of the imagined measurements, while \hbar denotes the Planck's constant.

Relation (2) was introduced in [27, 28] and it is depicted as above in terms of traditional QM notations [29, 30]. The main features of the respective notations are reminded briefly below in Appendices A and B while some aspects regarding the Dirac's bracket QM notations [29–32] are discussed in Appendix B.

Note here the fact that the right-hand side term from (2) is dependent on Planck's constant \hbar , e.g. $|\langle [\hat{A}, \hat{B}] \rangle_{\Psi}| = \hbar$ when A and B are (canonically) conjugate.

Starting from the generative elements (1) and (2), CIUR doctrine jointly with UR and QMS philosophy have been evolved around the following **Basic Precepts (BP)**:

- **BP₁**: Quantities Δ_{TEA} and $\Delta_{\Psi A}$ from relations (1) and (2), have similar significances of measuring uncertainties for the observable A . Consequently, the respective relations should be regarded as having a same meaning of Uncertainty Relations (UR) concerning the simultaneous measurements of observables A and B . Such a regard is fortified much more by the fact that

$$\left| \langle [\hat{A}, \hat{B}] \rangle_{\Psi} \right| = \hbar$$

when A and B are (canonically) conjugate.

- **BP₂**: In case of a solitary observable A , for a micro-particle, the quantities Δ_{TEA} or $\Delta_{\Psi A}$ can have always an unbounded small value. Therefore such an observable should be considered as measurable without any uncertainty in all cases of micro-particles (systems) and states.
- **BP₃**: For two commuting observables A and B (whose operators \hat{A} and \hat{B} commute, i.e. $[\hat{A}, \hat{B}] = 0$) relation (2) allows for the product $\Delta_{\Psi A} \cdot \Delta_{\Psi B}$ to be no matter how small. Consequently the quantities $\Delta_{\Psi A}$ and $\Delta_{\Psi B}$ can be unlimited small at the same time. Such observables have to be regarded as being compatible, i.e. measurable simultaneously and without interconnected uncertainties, for any micro-particle (system) or state.
- **BP₄**: In case of two non-commuting observables A and B (described by operators \hat{A} and \hat{B} which do not commute, i.e. $[\hat{A}, \hat{B}] \neq 0$) the relation (2) shows that the product $\Delta_{\Psi A} \cdot \Delta_{\Psi B}$ has as lower bound a non-null and \hbar -dependent quantity. Then the quantities $\Delta_{\Psi A}$ and $\Delta_{\Psi B}$ can be never reduced concomitantly to null values. For that reason the respective observables must be accounted as measurable simultaneously only with non-null and interconnected uncertainties, for any situation (particle/state). Viewed in a pair such observables are proclaimed as being incompatible, respectively complementary when they are (canonically) conjugate.

- **BP₅**: The main elements of CIUR doctrine and UR philosophy show quantum particularities of uniqueness comparatively with other non-quantum areas of physics. Such elements are the very existence of relations (1) and (2), the above asserted measuring features and the discriminating presence of the Planck's constant \hbar .
- **BP₆**: For glorifying the precepts **BP₁–BP₅** and adopting the usages of dominant literature, UR philosophy in its entirety should be ranked to a status of fundamental concept named Uncertainty Principle (UP).

Add here the observation that, in their wholeness, CIUR doctrine conjointly with UR and QMS prevalent philosophy emerge completely from the assertions embedded in basic precepts “**BP₁–BP₆**”.

3 Deficiencies (D) of the mentioned precepts

The above mentioned emergence conceals a less popularized fact namely that each of the precepts **BP₁–BP₆** is discredited (and denied) by insurmountable deficiencies. Such a fact can be revealed through a deep analysis of the respective precepts, an analysis which is of major importance for an authentic and fruitful survey of UR and QMS prevalent philosophy. That is why here below we aim to reveal the most significant ones of the mentioned deficiencies. They will be presented in a meaningful ensemble, able to give an edifying global appreciation regarding the mentioned philosophy. The referred ensemble includes as distinct pieces the following *Deficiencies (D)*:

3.1 *D₁*: Provisional character of relation (1)

Now it must be noted firstly the aspect that, through an analysis of its origins, relation (1) shows only a provisional (transient) character. This because it was founded [25, 26] on old resolution criterion from optics (introduced by Abe and Rayleigh — see [33]). But the respective criterion was surpassed through the so-called super-resolution techniques worked out in modern experimental physics (see [34–38] and references). Then by means of of the mentioned techniques can be imagined some interesting “*Super-Resolution-Thought-Experiments*” (*SRTE*). Through such *SRTE* for two (canonically) conjugate observables A and B , instead of *TE*-uncertainties $\Delta_{TE}A$ and $\Delta_{TE}B$ from (1), it becomes possible to discuss situations with some *SRTE*-uncertainties denoted as $\Delta_{SRTE}A$ and $\Delta_{SRTE}B$. For the respective *SRTE*-uncertainties, instead of Heisenberg's restrictive formula (1) (first version), can be suggested some CIUR-discordant relations like as

$$\Delta_{SRTE}A \cdot \Delta_{SRTE}B < \hbar. \quad (3)$$

Note that an experimental example of discordant relation of (3)-type was mentioned in [39] (where the UR (1) “*would be violated by close to two orders of magnitude*”).

Now one observes that, from the our days scientific perspective, *SRTE* relations like (3) are suitable to replace the

old Heisenberg's formula (1) (second version). But such suitability invalidates a good part of the precept **BP₁** and, additionally, it incriminates the CIUR doctrine and UR–QMS philosophy in connection with one of their main (generative) element.

It is surprising that, after invention of the super-resolution techniques, the mainstream (normative /authoritarian) publications connected with UR–QMS philosophy avoided a just and detailed evaluation of the respective techniques. Particularly, even after eight year after the result reported in [39], almost all of the dominant publications omit to discuss the respective result. The surprise is evidenced to a great extent by the fact that parsimony desiderata noted in Section 1 offer a viable argumentation for completing the evaluations and discussions off the mentioned kind.

Another infringement (violation) of Heisenberg's relation (1) was reported in [40] as an experimental result. That report is criticized vehemently by CIUR partisans [12]. The respective criticism is done in terms of a few un-argued (and un-explained) accusatory-sentences. But it is expected that, if they are justifiable, such kind of critiques should be grounded on precise technical details and arguments. This in order that they to be credible.

Curiously is also the fact that, over the past decades within the UR philosophy, the debates have neglected the older criticisms of the relation (1) due to K. Popper [41].

Taking into account the above revealed aspects one can say that the precept **BP₁** proves oneself to be a misleading (even harmful) basic element for CIUR doctrine and UR–QMS philosophy. But such a proof is a first argument for reporting that the respective doctrine and philosophy cannot be accepted as solid (and credible) scientific constructions.

3.2 *D₂*: Significance of quantities from relation (2)

The term “*uncertainty*” used within CIUR doctrine for quantities $\Delta_{\Psi}A$ and $\Delta_{\Psi}B$ from (2) is groundlessly because of the following considerations. According the theoretical framework of QM, by their definitions, the respective quantities signify genuinely the standard deviations of the observables A and B regarded as random variables (see below Appendix A). With such significances the alluded quantities refer to intrinsic (own) properties (known as fluctuations) of the considered particle but not to characteristics of the measurements performed on respective particle. In fact, on a one hand, for a measured particle in a given state (described by certain wave function Ψ) the quantities $\Delta_{\Psi}A$ and $\Delta_{\Psi}B$ have unique and well definite values. On the other hand for the same particle/state the measuring uncertainties regarding the observables A and B can be changed through the improvements or deterioration of experimental devices/techniques.

The above revealed QM significances for quantities $\Delta_{\Psi}A$ and $\Delta_{\Psi}B$ are genuinely preferable comparatively with the assertions from the precepts **BP₁–BP₄** promoted by CIUR doc-

trine and UR–QMS philosophy. But such a preference is completely congruent with the previously mentioned desiderata of parsimony principle.

3.3 D_3 : Limitations of relation (2)

Relation (2) has only limited validity within the complete theoretical framework of QM. This because, as it is detailed below in Appendix A, for observables A and B , relation (2) is only a restricted consequence of the generally valid Cauchy-Schwarz formula, given in (A.2). From such a general formula the relation (2) results iff (if and only if) in circumstances when the conditions (A.3) are satisfied. In the respective circumstances in addition to relation (2)/ (A.7) from (A.2) arises also the formula (A.6). It is worthy to note that the mentioned particularities regarding the validity of the relation (2) discredit indirectly the precept BP_1 of CIUR doctrine and UR–QMS philosophy. In their essence the specifications recorded here are nothing but concretizations of parsimony desiderata regarding the respective doctrine and philosophy.

3.4 D_4 : On solitary observables

It is surprising to find that, within UR–QMS philosophy debates, the problem of solitary observables is not discussed carefully. Particularly, were neglected discussions regarding the measurements of such observables. This although the respective discussions can be sub-summed to the question of simultaneous measurements of two observables. Such a sub-summation can be imagined by means of the Thought Experiments (TE) which motivated the conventional relation (1). Namely, for example, if in the respective TE it is of interest only the quantity $\Delta_{TE}A$, by ignoring completely the quantity $\Delta_{TE}B$, one can say that $\Delta_{TE}A$ can be unlimited small. Therefore the observable A , regarded as a solitary variable, appears as measurable without any uncertainty in all cases. But, on the other hand, if the same solitary observable A is analyzed in terms of relation (2), it cannot be associated with an unlimited small value for the quantity $\Delta_{\Psi}A$. This because, from a QM perspective, $\Delta_{\Psi}A$ has a unique and well definite value, evaluated through the corresponding wave function Ψ . Consequently, even in the cases of solitary observables, the CIUR doctrine and the UR–QMS philosophy cannot provide a clear and unequivocal approach as it is suggested by precept BP_2 .

3.5 D_5 : About commutable observables

According to the precept BP_3 for two observables A and B , whose associated operators \hat{A} and \hat{B} are commutable, relation (2), allows for the product $\Delta_{\Psi}A \cdot \Delta_{\Psi}B$ to be however small. Then the quantities $\Delta_{\Psi}A$ and $\Delta_{\Psi}B$ can be unlimited small at the same time. Such observables are supposed compatible, they being measurable simultaneously and without interconnected uncertainties for any micro-particle (system) or state.

But, as it was shown above in deficiency D_2 , the mentioned assertions from BP_3 , conflict with the genuine signifi-

cance of the quantities $\Delta_{\Psi}A$ and $\Delta_{\Psi}B$. This because both $\Delta_{\Psi}A$ and $\Delta_{\Psi}B$ have unique values, determined theoretically by the wave function Ψ which describe the considered state of particle. Or it is possible to have some “rebellious situations” in which the respective values of $\Delta_{\Psi}A$ and $\Delta_{\Psi}B$ to be simultaneously non-zero but finite entities, even the corresponding observables are commutable.

Such a “rebellious situation” can be found [20] for the observables P_x and P_y (Cartesian moments) regarding a micro-particle situated in a potential well of a rectangular 2D configuration. If the well walls are inclined towards the X and Y axes, the both the quantities $\Delta_{\Psi}P_x$ and $\Delta_{\Psi}P_y$ have non-zero but finite values. In that situation for P_x and P_y , besides the relation (2), it is satisfied however the formula (A.2) with $|(\delta_{\Psi}\hat{P}_x\Psi, \delta_{\Psi}\hat{P}_x\Psi)|$ as a non-null quantity.

The above remarks show that, in fact, the cases of commutable observables require to repudiate firmly the precept BP_3 . Additionally we think that the same cases should be regarded in the spirit of parsimony principle desiderata, by their consideration in QM terms reminded briefly in Appendices A and B.

3.6 D_6 : Cases of angular observables L_z and φ

The precept BP_4 stipulates that, as a principle, two non-commutable observables A and B cannot be measured simultaneously because the product $\Delta_{\Psi}A \cdot \Delta_{\Psi}B$ has a non-null lower bound. But the respective stipulation is contradicted by some rebellious pairs of observables. Such a pair, widely discussed, is L_z – φ (angular momentum — azimuthal angle), regarded in certain particular situations. The respective contradiction was probably the most inciting subject of debates during the history of CIUR doctrine and UR–QMS philosophy (see [5, 17–20, 42–55]). The mentioned debates regarded mainly the quantum rotations which can be called “ L_z -non-degenerate — circular — rotations” (L_z -ndcr). But, besides of that situations, in QM framework can be discussed also other kinds of rotations, of direct significance for L_z – φ pair. Such kinds are the ones regarding the rotational eigenstates of a Quantum Torsion Pendulum (QTP) and respectively the “ L_z -degenerate — spatial — rotations” (L_z -dsr). The true situations of the L_z – φ pair in relation with all kinds of the mentioned rotations will be discussed below in more details.

3.6.1 D_{6a} : About non-degenerate circular rotations

Let us discuss now the cases of L_z -non-degenerate — circular — rotations (L_z -ndcr). As systems of with L_z -ndcr can be quoted the following ones: (i) a particle (bead) on a circle, (ii) an 1D rotator and (iii) non-degenerate spatial rotations of a particle on a sphere or of an electron in a hydrogen atom respectively. The mentioned spatial rotations are considered as L_z -non-degenerate if the magnetic quantum number m (associated with L_z) has a unique value (while, of course, all other specific (orbital) quantum numbers have well-defined values).

The rotations of respective systems are described through the wave functions given by

$$\Psi(\varphi) = \Psi_m(\varphi) = (2\pi)^{-\frac{1}{2}} \cdot \exp(im\varphi). \tag{4}$$

Here φ is an ordinary polar coordinate (angle) with the corresponding mathematical characteristics [56] i.e. $\varphi \in [0, 2\pi)$ and number m gets only one value from the set $m = 0, \pm 1, \pm 2, \dots$. Also in (4) the wave function $\Psi(\varphi) = \Psi_m(\varphi)$ has the property $\Psi(0) = \Psi_m(2\pi - 0) := \lim_{\varphi \rightarrow 2\pi-0} \Psi_m(\varphi)$.

In the same context, according to the known QM framework [29], L_z and φ should be regarded as polar observables, described by the conjugated operators and commutator represented as follows

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}, \quad \hat{\varphi} = \varphi, \quad [\hat{L}_z, \hat{\varphi}] = -i\hbar. \tag{5}$$

Therefore the conventional relation (2) motivates as a direct consequence the next formula

$$\Delta_\Psi L_z \cdot \Delta_\Psi \varphi \geq \frac{\hbar}{2}. \tag{6}$$

Now it is easy to observe that this last formula is explicitly inapplicable in cases described by wave functions (4). This because in such cases, for the quantities $\Delta_\Psi L_z$ and $\Delta_\Psi \varphi$ associated with the pair $L_z-\varphi$, one obtains the following values

$$\Delta_\Psi L_z = 0, \quad \Delta_\Psi \varphi = \pi \cdot (3)^{-\frac{1}{2}}. \tag{7}$$

But such values for $\Delta_\Psi L_z$ and $\Delta_\Psi \varphi$ are evidently incompatible with the conventional relation (2)/(6).

In order to avoid the above revealed incompatibility in many mainstream publications the CIUR partisans promoted some unusual ideas such are:

- For L_z and φ operators and commutator, instead of current expressions (5), it is conveniently to adopt other new denotations (definitions).
- The formula (6) must be abandoned/proscribed and replaced by one (or more) “modified $L_z-\varphi$ UR” able to mime the conventional relation (2) for the $L_z-\varphi$ pair.

The alluded ideas were promoted through the conception of “impossibility of distinguishing ... between two states of angle differing by 2π ”. But such a conception has not any realistic sense in cases of circular rotations. This because in such cases the angle φ has as physical range the interval $[0, 2\pi)$. Moreover in the respective cases the wave functions (4) are normalized on the same interval but not on other strange domains.

As regards the “modified $L_z-\varphi$ UR”, along the years, by means of some circumstantial (and more or less fictitious) considerations, were proposed a lot of such relations. In terms of usual QM notations (summarized below in Appendix A),

the alluded “modified $L_z-\varphi$ UR” can be written generically as follows

$$f(\Delta_\Psi L, \Delta_\Psi g(\varphi)) \geq \hbar \cdot \langle s(\varphi) \rangle_\Psi. \tag{8}$$

Here $f(\Delta_\Psi L, \Delta_\Psi g(\varphi))$, $g(\varphi)$ and $s(\varphi)$ denote some specially invented functions depending on the corresponding arguments. Note that some of the mostly known concrete examples of relations (8) can be found collected in [55].

Now it should be noted the fact that the “modified $L_z-\varphi$ UR” such are (8) show some troubling features like the following ones:

- Regarded comparatively, the mentioned “modified $L_z-\varphi$ UR” are not mutually equivalent. This despite of the fact that they were invented in order to substitute the same proscribed formula (6). Consequently, none of that modified relations, is agreed unanimously as a suitable model able to give such a substitution.
- Relations (8) are in fact ad hoc artifices without any source in mathematical framework of QM. Then, if one wants to preserve QM as a unitary theory, like it is accredited in our days, the relations (8) must be regarded as unconvincing and inconvenient (or even prejudicial) inventions.
- In fact in relations (8) the relevant angular quantities $\Delta_\Psi L_z$ and $\Delta_\Psi \varphi$ are substituted more or less factitious with the adjusting functions $f(\Delta_\Psi L_z, \Delta_\Psi g(\varphi))$, $g(\varphi)$ and $s(\varphi)$. But, from a genuine perspective, such substitutions, and consequently the corresponding relations, are only mathematical constructs but not elements with useful physical significance. Of course that such constructs overload (or even impede) the scientific discussions by additions of extraneous entities which are not associated with true information about the real world.

Then, for a correct evaluation of the facts, all the aspects regarding relations (8) versus (6) ought to be judged by taking into consideration the parsimony principle desideratum: “Entities are not to be multiplied beyond necessity”. Such an evaluation can be started by clarifying firstly the origin and validity conditions of the formula (6) regarded as descendant of conventional relation (2). For the respective clarification it is usefully to see some QM elements briefly summarized in Appendix A.

So it can be observed easy that, in its essence, the relations (2) follow from the generally valid formulas (A.2) pertaining to the mathematical framework of QM. But, attention, (2) results correctly from (A.2) iff (if and only if) when it is satisfied the condition (A.3). In other cases (2) are not valid at all. Such an invalidity is completely specific for the cases of $L_z-\varphi$ pair in relations with situations described by the wave functions (4). This because in respective cases instead of conditions (A.3) it is true the relation

$$(\hat{L}_z \Psi, \hat{\varphi} \Psi) = (\Psi, \hat{L}_z \hat{\varphi} \Psi) + i\hbar. \tag{9}$$

Therefore, for systems described by the wave functions (4), the formula (6) is invalid by its essence.

Now note that, even when the condition (A.3) is not satisfied, according to the QM general formula (A.2), for the discussed situations it is true the relation

$$\Delta_{\Psi} L_z \cdot \Delta_{\Psi} \varphi \geq \left| \left(\delta_{\Psi} \hat{L}_z \Psi, \delta_{\Psi} \hat{\varphi} \Psi \right) \right| \quad (10)$$

written in compliance with definitions (4) and (5). But, attention, in respective situations the last relation (10) degenerates into trivial equality “ $0=0$ ”. Add here the fact that relation (10) is completely equivalent with the formula (C.13) deductible within Fourier analysis.

The above presented details argue undoubtedly the view that in cases with L_z -ndcr the L_z - φ pair must to satisfy not the troublesome formula (6) but the QM justified relation (10) (which in fact reduces itself to banal equality “ $0=0$ ”). Such an argued view clarifies all disputes regarding the mentioned cases. Moreover the same view disproves the idea of some “entities ... multiplied beyond necessity” (such are the modified UR (8)) intended to replace the inoperative relation (6).

3.6.2 D_{6b} : Case of Quantum Torsion Pendulum (QTP)

The case of Quantum Torsion Pendulum (QTP) regards a quantum harmonic oscillator with torsional rotations [19, 20, 55]. Such an oscillator can be considered as the simplest theoretical model for molecular twisting motion (“change in the angle between the planes of two groups of atoms” [57]). For a QTP oscillating around the z-axis the Hamiltonian operator has the form

$$\hat{H} = \frac{1}{2I} \hat{L}_z^2 + \frac{1}{2} I \omega_0^2 \varphi^2. \quad (11)$$

Here φ denotes the twisting angle with domain $\varphi \in (-\infty, +\infty)$ while the operators \hat{L}_z and $\hat{\varphi}$ obey the rules (5). The other symbols from (11) are: I and ω_0 represent the momentum of inertia respectively the (undamped) resonant frequency ($\omega_0 = \sqrt{\kappa/I}$ while κ = torsion elastic modulus).

By means of Schrödinger equation $E\Psi = \hat{H}\Psi$ one finds that the QTP eigenstates are described by the wave functions

$$\Psi_n(\varphi) = \Psi_n(\xi) \propto \exp\left(-\frac{\xi^2}{2}\right) \cdot \mathcal{H}_n(\xi), \quad \xi = \varphi \sqrt{\frac{I\omega_0}{\hbar}}. \quad (12)$$

These wave functions correspond to the oscillation quantum numbers $n = 0, 1, 2, 3, \dots$ and energy eigenvalues $E_n = \hbar\omega_0\left(n + \frac{1}{2}\right)$. In (12) $\mathcal{H}_n(\xi)$ represent the Hermite polynomials of ξ .

For each of the states (12) for observables L_z and φ associated with the operators (5) one obtains the expressions

$$\begin{aligned} \Delta \varphi &= \sqrt{\frac{\hbar}{I\omega_0} \left(n + \frac{1}{2}\right)}, & \Delta L_z &= \sqrt{\hbar I \omega_0 \left(n + \frac{1}{2}\right)}, \\ \left| \left(\Psi, \left[\hat{L}_z, \hat{\varphi} \right] \right) \right| &= \hbar, \\ \Delta \varphi \cdot \Delta L_z &= \hbar \cdot \left(n + \frac{1}{2}\right). \end{aligned} \quad (13)$$

These expressions show the fact that, for each QTP eigenstate, the L_z - φ pair satisfies the relation (6)/(2). But note that the respective fact is due to the circumstance that in the mentioned case, in relation with the wave functions (12), the operators \hat{L}_z and $\hat{\varphi}$ satisfy a condition of (A.3) type, i.e. $(\hat{L}_z \Psi, \hat{\varphi} \Psi) = (\Psi, \hat{L}_z \hat{\varphi} \Psi)$.

3.6.3 D_{6c} : On degenerate spatial rotations

Let us now regard the cases of L_z -degenerate-spatial-rotations (L_z -dsr). Such kinds of rotations refer [20, 21, 55] to states of: (i) a particle on a sphere, (ii) a 2D rotator and (iii) an electron in a hydrogen atom. The respective rotations are L_z -degenerate in sense that the magnetic quantum number m (associated with L_z) has multiple values while the other quantum numbers have unique values. A particle on a sphere or a 2D rotator are in a L_z -dsr when the orbital number l has a unique value greater than zero while m can take all the values $m \in [-l, +l]$. Then the corresponding rotations are described through the global wave function

$$\Psi(\vartheta, \varphi) = \Psi_l(\vartheta, \varphi) = \sum_{m=-l}^{m=+l} c_m \cdot Y_{lm}(\vartheta, \varphi). \quad (14)$$

Here ϑ and φ denote polar respectively azimuthal angles with $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. In (14) $Y_{lm}(\vartheta, \varphi)$ denote spherical functions while c_m are coefficients normalized through the condition $\sum_{m=-l}^{m=+l} |c_m|^2 = 1$. Also the wave functions $\Psi_l(\vartheta, \varphi)$ from (14) have the property $\Psi_l(\vartheta, 0) = \Psi_l(\vartheta, 2\pi - 0) := \lim_{\varphi \rightarrow 2\pi-0} \Psi_l(\vartheta, \varphi)$. In a direct connection with such a property the operators \hat{L}_z and $\hat{\varphi}$ obey the rules (5).

Now let us regard what are the peculiarities of the L_z -dsr cases in respect with the controversial relation (6). Principled, such a regard demands that, by using the formulas (5) and (14), to evaluate the corresponding expressions for the quantities $\Delta_{\Psi} L_z$, $\Delta_{\Psi} \varphi$ and $\left| \left(\Psi, \left[\hat{L}_z, \hat{\varphi} \right] \Psi \right) \right|$. With the respective expressions one finds possibilities that the relation (6) to be or not to be satisfied. Of course that such possibilities are conditioned by the concrete values of the coefficients c_m . But note that, if the relation (6) is not satisfied, the fact appears because essentially in such a situation the condition (A.3) is not fulfilled. Add here the important observation that, independently of validity for relation (6), in all cases of L_z -dsr the L_z - φ pair obeys the prime QM relation (A.2) through adequate values for the quantities $\Delta_{\Psi} L_z$, $\Delta_{\Psi} \varphi$ and $\left| \left(\delta_{\Psi} \hat{L}_z \Psi, \delta_{\Psi} \hat{\varphi} \Psi \right) \right|$. The previous considerations offer a clear evaluation of the situation for L_z -dsr cases relatively to the conventional relation (2) and precept BP_4 .

Summing up of deficiencies D_6 (including D_{6a} , D_{6b} and D_{6c}): The above discussion about the three kinds of rotations reveals the deficiencies of the conventional relation (2) and of the associated precept BP_4 in regard with the non-commutable observables L_z and φ . But such revealing is noth-

ing but a direct and irrefutable incrimination of CIUR doctrine and UR-QMS philosophy.

3.7 D₇: On number and phase observables

The pair N and ϕ (number and phase) is another couple of rebellious non-commutable observables which contradict the corresponding stipulation from the precept **BP₄** of UR-QMS philosophy. That contradiction emerged in connection with the associated operators \hat{N} and $\hat{\phi}$. The respective operators were introduced by means of the ladder (lowering and raising) operators \hat{a} and \hat{a}^+ , destined to convert some QM calculations procedures from an analytical version to an algebraic one. Through the respective connection, by taking as base the relation $[\hat{a}, \hat{a}^+] = 1$, it was inferred the commutation formula $[\hat{N}, \hat{\phi}] = i$.

The last noted formula motivated the idea that operators \hat{N} and $\hat{\phi}$ must satisfy the conventional relation (2) with both $\Delta_\Psi N$ and $\Delta_\Psi \phi$ as non-null quantities. But afterward it was found the fact that, in the case of a harmonic oscillator eigenstates, one obtains $\Delta_\Psi N = 0$ and $\Delta_\Psi \phi = \pi \cdot (3)^{-\frac{1}{2}}$ i.e. a violation of the relation (2). Of course that such a fact leads to a deadlock for harmonization of N - ϕ observables with the CIUR doctrine and UR-QMS philosophy. Note that this deadlock is completely analogous with the one regarding to L_z - φ observables in the above discussed case of L_z - $ndcr$ (L_z -non-degenerate — circular — rotations).

For avoiding the mentioned N - ϕ deadlock in many publications were promoted various adjustments (see [6, 43, 48, 58–61] and references therein). But it is easy to observe that the respective adjustments regarded the conventional relation (2) as an absolute mark and tried to adapt accordingly the pair N - ϕ for a description of a harmonic oscillator. So it was suggested to replace the original operators \hat{N} - $\hat{\phi}$ by some ad hoc “adjusted” (*adj*) operators \hat{N}_{adj} and $\hat{\phi}_{adj}$, able to generate formulas resembling (more or less) with the conventional relation (2) (examples of such adjusted operators can be found in the literature of recent decades). However it is very doubtfully that the corresponding “adjusted observables” N_{adj} and ϕ_{adj} can have natural (or even useful) physical significances. Moreover, until now, it not exist a unanimously agreed conception able to guarantee a true elucidation regarding the status of number-phase observables relatively to terms of CIUR doctrine and UR philosophy.

Our opinion is that a genuine clarification of the N - ϕ problem can be done similarly with the above discussed situation of L_z - φ observables in the cases of L_z - $ndcr$. More exactly we have to note that the disagreement of N - ϕ pair with the conventional relation (2) results from fact that in such a case the respective relation is mathematically incorrect. The aforesaid incorrectness is due mainly to the circumstance that, in cases of a linear oscillator eigenstates, the N - ϕ pair does not satisfy the essential condition (A.3). This because in that cases for the operators \hat{N} - $\hat{\phi}$ is true the for-

mula $(\hat{N}\Psi, \hat{\phi}\Psi) = (\Psi, \hat{N}\hat{\phi}\Psi) + i$ which evidently infringes the condition (A.3). But it should be pointed out that, even in the mentioned cases, the \hat{N} - $\hat{\phi}$ operators satisfy the primary relation (A.2) which degenerates into trivial equality “ $0 = 0$ ”.

We think that the above noted opinion gives a natural and incontestable solution for the problem regarding the N - ϕ pair versus the conventional relation (2). Accordingly the fictional operators \hat{N}_{adj} and $\hat{\phi}_{adj}$, of an ad hoc adjusted essence, proves themselves to be nothing but “*entities . . . multiplied beyond necessity*”.

So it can be said that the situation of observables N and ϕ contradict directly the precept **BP₄** in connection with non-commutable observables. Consequently, the respective situation invalidates completely one of basic elements of CIUR doctrine and UR-QMS philosophy.

3.8 D₈: Concerning the energy — time pair

Closely to the conventional views of CIUR doctrine and UR-QMS philosophy the pair of observables E - t (energy-time) was subject for a large number of controversial discussions (e.g. in works [5, 6, 62–64], in their references and, certainly, in many other publications). The alluded discussions were generated within following circumstances. On one hand, accordingly to the mentioned views, E and t are regarded as conjugated observables, having to be described by the next operators and commutator

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \quad \hat{t} = t, \quad [\hat{E}, \hat{t}] = i\hbar. \tag{15}$$

Then the operators \hat{E} and \hat{t} should satisfy the conventional relation (2) in a nontrivial version. On the other hand, because of the fact that, in terms of usual QM, the time t is a deterministic but not random variable, for any quantum situation one finds the following expressions $\Delta_\Psi E = \text{“a finite quantity”}$ respectively $\Delta_\Psi t \equiv 0$. But these expressions invalidate the relation (2) and consequently the E - t pair shows an anomaly in respect with the alluded conventional ideas, especially with the precept **BP₄**. For avoiding the noted anomaly, within the literature about E - t pair, it was substituted the unsuitable relation (2) by some adjusted formulas written generically as follows

$$\Xi E \cdot \Xi t \geq \frac{\hbar}{2}. \tag{16}$$

The so introduced quantities ΞE and Ξt have various significances such are: (i) line-breadth and half-life of a decaying excited state, (ii) frequency domain and temporal widths of a wave packet, (iii) $\Xi E = \Delta_\Psi E$ and $\Xi t = \Delta_\Psi A \cdot (d(A)/dt)^{-1}$, with A = an arbitrary observable.

As regards the adjusted formulas (16) note firstly the fact that various of their versions are not congruent with the original conception of relation (2). Also the respective versions are not mutually equivalent from a mathematical (theoretical) viewpoint. So they have no reasonable justification in

the true QM framework. Moreover in specific literature none of the formulas (16) is accepted unanimously as a correct (or natural) substitute for conventional relation (2).

Now it is the place to present the following clarifying remarks. Even if the $E-t$ pair is considered to be described by the operators (15), according to the true QM terms, one finds the relation

$$(\hat{E}\Psi, \hat{t}\Psi) = (\Psi, \hat{E} \hat{t}\Psi) - i\hbar. \quad (17)$$

By comparing this relation with condition (A.3) one sees directly that the $E-t$ pair cannot ever satisfy the respective condition. This is the essential reason because of which for the $E-t$ pair the conventional relation (2) is not applicable at all. Nevertheless, for the same pair described by the operators (15), the QM relation (A.2) is always true. But because in QM the time t is a deterministic (i.e. non-stochastic) variable in all cases the respective true relation degenerates into the trivial equality " $0 = 0$ ".

The above noted comments lead to the next findings:

- In case of the $E-t$ pair the conventional views (of CIUR doctrine and UR-QMS philosophy) are completely nonfunctional.
- Genuinely, within a true QM framework, the time t is in fact a pure deterministic (non-stochastic) quantity without any standard deviation (or fluctuation).

But, taken together, such findings about time-energy pair must be reported as a serious and insurmountable deficiency of CIUR doctrine and UR-QMS philosophy.

3.9 D_9 : Atypical analogues of UR (1) and (2)

By basic precept **BP**₅ the UR philosophy claims idea that relations (1) and (2) possess an essential typicality represented by their QM uniqueness related with the systems of atomic size. Consequently, the respective relations should not have analogues in other areas of physics or for systems of radically different sizes. But the respective idea is definitely denied by some example that we will present below.

3.9.1 D_{9a} : Classical Rayleigh formula

As a first example of an atypical analogue of the UR (1) can be quoted the formula

$$\sin \alpha \cong \frac{\lambda}{d} \quad (18)$$

which expresses [35, 39, 40, 65] the Rayleigh resolution criterion from classical optics. In (18) α denotes the "angular resolution", λ is the wavelength of light, and d represents the diameter of lens aperture. Note that criterion (18) was introduced in classical optics in 1879, i.e. by long time before the QM appeared. Later one relation (1) was introduced by taking in (18) $d \sim \Delta_{TE} \cdot q$ for coordinate uncertainty, respectively $\lambda = (\hbar/p)$ for momentum p (through wave-particle duality formula) and $p \cdot \sin \alpha \sim \Delta_{TE} \cdot p$ for momentum uncertainty.

3.9.2 D_{9b} : Classical "Gabor's uncertainty relation"

An example of an atypical analog of (2) can be found within the mathematical harmonic analysis in connection with a pair of random quantities regarded as Fourier conjugated variables (see [66, 67] and the Appendix C below). In non-quantum physics such an analogue is known [67] as "Gabor's uncertainty relation" which can be represented through the relation

$$\Delta t \cdot \Delta \nu \geq \frac{1}{4\pi}. \quad (19)$$

This last relation (19) shows the fact that for a classical signal, regarded as a wave packet (of acoustic or electromagnetic nature), the product of the "uncertainties" ("irresolutions") Δt and $\Delta \nu$ in the time and frequency domains cannot be smaller than a specific constant.

3.9.3 D_{9c} : A relation regarding thermodynamic observables

Another example of an atypical similar of UR (2) is given by the following classical formula

$$\Delta_W \mathbb{A} \cdot \Delta_W \mathbb{B} \geq |\langle \delta_W \mathbb{A} \cdot \delta_W \mathbb{B} \rangle_W| \quad (20)$$

showed as relation (D.3) in Appendix D of the present article. The elements (notations and physical significances) implied in (20) are those detailed in Appendix D. The respective elements are specific to the phenomenological theory, initiated by Einstein, about fluctuations of macroscopic thermodynamic observables (see [20, 68–72] and Appendix D below).

Note that, from the perspective of mathematics (more exactly of probability theory), the macroscopic formula (20) and UR (2) are analogue relations, both of them regard the fluctuations of the corresponding observables judged as random variables. Moreover they describe the intrinsic properties of considered systems (of macroscopic-thermodynamic respectively quantum nature) but not aspects of measurements performed on the respective systems. The corresponding measurements can be described through a distinct approaches modeled/depicted as transmission processes for stochastic data (see below Appendix E and Section 5 in present article).

As regards the formula (20), the following notifications should be done too. To a some extent the respective formula can be considered as being member to a family of so called "thermodynamic UR", discussed in a number of publications from the last century (see [78, 79] and references). Note that the alluded membership is true only in respect with the "regular" subset of respective family, derivable from the Einstein's phenomenological theory. But the mentioned family includes moreover a class of "irregular" relations. The most known such an "irregular" relation regards the conjugate variables energy U and temperature T of a thermodynamic system. It

has [78] the form

$$\Delta U \cdot \Delta \left(\frac{1}{T} \right) \geq k_B \tag{21}$$

where k_B denote the Boltzmann’s constant.

It must be noted now the reality that fluctuation formula (20) and “irregular” relations like is (21) are completely dissimilar, first of all, due to the important distinction between reference frames of their definitions. The respective dissimilarity is pointed out by the following aspects. On the one hand, the quantities $\Delta_W \mathbb{A}$ and $\Delta_W \mathbb{B}$ from (20) are defined by referring to the same state of the considered system. On the other hand the quantities U and T which appear in (21) refer to different states of a system, namely states characterized by an energetic isolation respectively by a thermal contact. Due mainly to the above mentioned dissimilarity: “*a derivation of the uncertainty relation (21) analogous to that of the usual Heisenberg relations (i.e. UR (2)) is impossible*” [78].

Add here the fact that, within associate literature, it was reported a number of controversies about the aspects regarding the possible similarities between the “*thermodynamic UR*” (mainly from the same subset as (21)) and quantum UR (2) (see [78] and references). Among respective aspects can be quoted:

- compatibility of macroscopic observables,
- commutativity of thermodynamic variables and
- reconstruction of QM from hidden variables theories similarly with the rebuilding of thermodynamics through subjacent molecular considerations.

Note that the just mentioned aspects are not taken into account (as relevant elements) for our present survey on deficiencies of prevalent philosophy regarding UR and QMS.

3.9.4 D_{9d} : On the so called macroscopic operators

In the spirit of conventional precept BP_5 the uniqueness of UR (2) consists in its strict specificity for micro-particles (of atomic size), without analogues in cases of macroscopic systems. But, as it is pointed out through relation (D.12) from Appendix D, in case of macroscopic thermodynamic system studied in quantum statistical physics one finds the formula

$$\Delta_\rho \mathbb{A} \cdot \Delta_\rho \mathbb{B} \geq \frac{1}{2} \left| \left\langle \left[\hat{\mathbb{A}}, \hat{\mathbb{B}} \right] \right\rangle_\rho \right|. \tag{22}$$

This last formula is similar with the conventional UR (2) (more exactly, mathematically, with its primary versions (A.7) and (B.4)). Due to such a similarity, probably, some publications (e.g. [74] and references) have tried to regard (22) as a macroscopic UR. But the respective regard was found to be incompatible with the known UR–QMS philosophy, mainly with the precept BP_4 .

The alluded incompatibility is pointed out by the following facts. On the one hand, in spirit of UR philosophy (precepts BP_1 – BP_4), the quantities $\Delta_\rho \mathbb{A}$ and $\Delta_\rho \mathbb{B}$ from (22) should be considered as measuring uncertainties of macroscopic observables \mathbb{A} and \mathbb{B} . Additionally when the operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ and do not commute (i.e. $[\hat{\mathbb{A}}, \hat{\mathbb{B}}] \neq 0$), according to (22), the quantities $\Delta_\rho \mathbb{A}$ and $\Delta_\rho \mathbb{B}$ can be never reduced concomitantly to null values. Consequently, in terms of UR–QMS philosophy, for any situation, the non-commutable macroscopic observables \mathbb{A} and \mathbb{B} are allowed to be measurable simultaneously only with non-null and interconnected uncertainties. But, on the other hand, according to the classical physics any two macroscopic observables can be measured concurrently with unlimited accuracies and without any interrelated uncertainties.

For avoiding the above noted incompatibility some partisans of UR philosophy have suggested the following expedient. Abrogation of (22) by replacement of genuine macroscopic operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ with another quasi-diagonal operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ (i.e. with operators whose representations in any base are quasi-diagonal matrices). Such substituting operators should commute and so the right hand term in (22) to be (quasi) null (i.e. $\left| \left\langle \left[\hat{\mathbb{A}}, \hat{\mathbb{B}} \right] \right\rangle_\rho \right| \approx 0$). Through the mentioned substitution the inconvenient relation (22) could be changed with the more convenient formula

$$\Delta_\rho \mathbb{A} \cdot \Delta_\rho \mathbb{B} \geq \frac{1}{2} \left| \left\langle \left[\hat{\mathbb{A}}, \hat{\mathbb{B}} \right] \right\rangle_\rho \right| \approx 0. \tag{23}$$

Then it seems to be possible that the substituted macroscopic uncertainties $\Delta_\rho \mathbb{A}$ and $\Delta_\rho \mathbb{B}$ to be reduced simultaneously to arbitrarily small (even zero) values. Apparently, such a possibility should to harmonize the interpretation of the relation (23) with the concepts of classical physics.

However, in fact, the above mentioned harmonization is not possible and the suggested expedient is useless. This, at least, due to the following reasons:

- Firstly, the relations (22) cannot be abrogated/substituted if the entire mathematical framework of quantum statistical physics is not abrogated/substituted too.
- Secondly, in common practice of studies of quantum statistical systems (e.g. such are the ones investigated in [80, 81]) are used the genuine operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ but not the quasi-diagonal ones $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$.
- As a third reason, the following fact can be also noted. Even in certain situations when the original operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ are quasi-diagonal in the sense of the mentioned expedient, the relation (23) does not turn into a form having a null term in the right hand side. Such a situation can be found [20] in case regarding a macroscopic paramagnetic system made of a huge number of independent 1/2-spins. In such a case as macroscopic operators appear the Cartesian components \hat{M}_α ($\alpha = x, y, z$) of the system magnetization. Note that the

operators \hat{M}_α are quasi-diagonal in the sense required by the aforesaid expedient/substitution. But, for all that, the respective operators do not commute because $[\hat{M}_\alpha, \hat{M}_\beta] = i\hbar\gamma \cdot \epsilon_{\alpha\beta\mu} \cdot \hat{M}_\mu$ (γ = magneto-mechanical factor and $\epsilon_{\alpha\beta\mu}$ denotes the Levi-Civita tensor).

By taking into account the above pointed out deficiencies D_9 (including D_{9a} , D_{9b} , D_{9c} and D_{9d}) one may record the following conclusion. The relations (D.12)/(22) are relations regarding macroscopic areas of physics but not pieces which should be adapted to the requirements of prevalent philosophy about UR and QMS.

3.10 D_{10} : On the uniqueness of quantum measurements

Let us refer now to the uniqueness character of conventional relations (1) and (2) with regard to the measurements peculiarities at quantum level. The aforesaid character was largely debated in literature and it has generated the still open questions about the main characteristics (conceptual relevance and description procedures) of Quantum Measurements (QMS). By promoting all the assertions from percepts BP_1 – BP_4 the UR–QMS philosophy tried to enforce the opinion that relations (1) and (2) are closely linked with the measuring particularities that are unique in quantum context, without any correspondence (analogy) in non-quantum domains of physics. The mentioned opinion, often promoted as a true dogma, dominates the mainstream of existing publications.

On the other hand, as we have argued above through the deficiencies D_1 – D_9 , the alluded opinion is completely unfounded because, genuinely, the respective relations are:

- either an old-fashioned (and removable) empirical convention (in case of (1)),
- or simple (non-magistral) theoretical formula (in case of (2)).

Within UR–QMS prevalent philosophy, as a widespread belief, the uniqueness peculiarities of QMS are motivated through the so called “observer effect”. The respective effect is presented as a perturbing influence of observer (by experimental devices) on investigated systems and on measuring results. It is presumed to differentiate radically the QMS from classical measurements (of macroscopic physics). Such effects are absolutely unavoidable and affected by notable uncertainties in quantum contexts but entirely preventable and with negligible inaccuracies in classical situations.

The above mentioned belief is categorically disproved by the following observations. The “observer effect” appear not only in QMS but also in some classical measurements (e.g. [82] in electronics or in thermodynamics). Of course that in classical cases the measuring inaccuracies can be made negligible (by adequate improvements of experimental devices and/or procedures). It should be noted, that, in principle, quantum uncertainties can be also diminished (for example, with the super-resolution techniques discussed above in D_1).

Then the idea of uniqueness quantum measuring character for conventional relations (1) and (2), promoted by UR philosophy through BP_5 , proves oneself as being a groundless fiction which should be disregarded. But such a disregard come to fortify the J. Bell’s thinking [83, 84] that: “*the word ‘measurement’ should be avoided (or even . . . banned) altogether in quantum mechanics*”. Some annotations about the respective thinking are given below in Section 5 where we will present briefly a non-conventional approach of QMS problems.

3.11 D_{11} : On the uniqueness of Planck’s constant

Another aspect of quantum uniqueness invoked in precept BP_5 regards the presence of Planck’s constant \hbar as a specific symbol in conventional quantum relations (1) and (2), comparatively with a total absence of some similar symbols in all classical (non-quantum) formulas. We shall examine the alluded aspect in regard with the relation (2). Then of prime importance is to notify the fact that, mathematically, quantum observables from the relation (2) have a stochastic (non-deterministic) character. But a completely similar character one finds in cases of macroscopic observables implied in formula (20) regarding fluctuations specific to macroscopic thermodynamic systems.

Both kinds of mentioned stochastic observables describe fluctuations (at quantum respectively macroscopic scale). The mentioned fluctuations are characterized quantitatively by the corresponding standard deviations such are $\Delta_\Psi A$ or $\Delta_W \mathbb{A}$. But, mathematically, the standard deviation indicates quantitatively the stochasticity (randomness) degree of an observable. This in the sense that the alluded deviation has a positive or null value as the corresponding observable is a random or, alternatively, a deterministic (non-stochastic) variable. Consequently the quantities $\Delta_\Psi A$ and $\Delta_W \mathbb{A}$ can be regarded as similar indicators of stochasticity for quantum respectively macroscopic observables.

In principle for macroscopic thermal fluctuations the standard deviations like is $\Delta_W \mathbb{A}$ can have various expressions (depending on system, state and observable). Apparently, it would seem that the respective expressions do not contain any common element. Nevertheless such an element can be found as being materialized by the Boltzmann’s constant k_B (see relation (D.4) in Appendix D below and articles [71, 73]). So, for any macroscopic fluctuating observable A , the quantity $(\Delta_W \mathbb{A})^2$ (i.e. dispersion = square of the standard deviation) appears as a product of Boltzmann’s constant k_B with factors which are independent of k_B .

This means that the quantity $(\Delta_W \mathbb{A})^2$, in its quality of quantitative indicator of thermal fluctuations, is directly proportional with k_B . Consequently $(\Delta_W \mathbb{A})^2$ has a non-null respectively null value as $k_B \neq 0$ or $k_B \rightarrow 0$ (Note that because k_B is a physical constant the limit $k_B \rightarrow 0$ means that the quantities directly proportional with k_B are negligible com-

paratively with other quantities of same dimensionality but independent of k_B). On the other hand, the standard deviation $\Delta_W \mathbb{A}$ is a particular indicator for macroscopic stochasticity revealed through thermal fluctuations.

Bringing together the above noted aspects it can be said that k_B has the qualities of an authentic generic indicator for thermal stochasticity which is specific for classical macroscopic fluctuating systems.

Now let us discuss about the quantum stochasticity whose particular indicators are the standard deviations $\Delta_\Psi A$. Based on the relations (13) one can say that in many situations the expressions for dispersions $(\Delta_\Psi A)^2$ consist in products of Planck constant \hbar with factors which are independent of \hbar . Then, by analogy with the above discussed macroscopic situations, \hbar places itself in the posture of generic indicator for quantum stochasticity.

The mentioned roles as generic indicators for k_B and \hbar (in direct connections with the quantities $\Delta_W \mathbb{A}$ and $\Delta_\Psi A$) regard the one-fold (simple) stochasticity, of thermal and quantum nature respectively. But in physics is also known a twofold (double) stochasticity, of a combined thermal and quantum nature. Such a kind of stochasticity one finds in cases of macroscopic thermodynamic systems composed of statistical assemblies of quantum micro-particles. The alluded twofold stochasticity can be evaluated in a way through the dispersions $(\Delta_\rho \mathbb{A}_j)^2$ which estimate the level of fluctuations in the mentioned systems (see [20, 73, 76] and Appendix D below). As it is noted in relation (D.13) the dispersions $(\Delta_\rho \mathbb{A}_j)^2$ can be given through of products containing the function $\mathfrak{f}(k_B, \hbar) = \hbar \cdot \coth\left(\frac{\hbar\omega}{2k_B T}\right)$ and factors which are independent of both k_B and \hbar .

Then it results that k_B and \hbar considered together turn out to be a couple of generic indicators for the twofold (double) stochasticity of thermal and quantum nature. Such a kind of stochasticity is significant or negligible in situations when $k_B \neq 0$ and $\hbar \neq 0$ respectively if $k_B \rightarrow 0$ and $\hbar \rightarrow 0$.

Now we can note the indubitable remark that Planck's constant \hbar has an authentic classical analog represented by the Boltzmann's constant k_B , both \hbar and k_B having relevant significances as generic indicators of stochasticity. But such an analogy contradicts directly the basic precept **BP**₅ of CIUR doctrine and UR-QMS philosophy.

3.12 D_{12} : On the excessive ranking of UR

The ranking of UR to a position of principle, is widespread in the dominant literature, mainly through the authoritative and normative writings of many leading scientists. Surprisingly the respective ranking is argued merely in few occasions (e.g. in [10]) but only partially and not convincingly.

However, in [10], it was signaled the fact that “*over the years, some authors and foremost K. Popper, have contested this view, of such a ... ‘ranking’*”. The mentioned contestation seems to have been motivated by the assertion: “un-

certainty relations cannot be granted the status of a principle on the grounds that they are derivable from the theory (‘QM’), whereas one cannot obtain the theory from the uncertainty relations”. The aforesaid motivation was minimized and repudiated [10] through of the conventional (and prevalent) opinion that: “*there are many statements in physical theories which are called principles even though they are in fact derivable from other statements in the theory in question*”. Note that in spite of the mentioned repudiation, it was added in [10] the noteworthy observation that “*Serious attempts to build up quantum theory as a full-fledged Theory of Principle on the basis of the uncertainty principle have never been carried out*”.

As regards the above presented controversy our belief can be expressed as follows. The Popper's contestation of UR ranking (i.e., in fact, of the precept **BP**₆) has a genuine character while the opposing conventional opinion is nothing but a questionable (and unfounded) attempt to preserve a predominant traditionalist doctrine (dogma).

Now, from another perspective, we wish to point out a new important aspect. On the one hand a true scientific conception attests indubitably the idea that: “*A principle is statement which is taken to be true at all times and all places where it is applicable*” [85]. On the other hand all previously proved deficiencies **D**₁–**D**₁₁ show that usual philosophy of UR is not valid in a wide class of situations where they should to be applied. Therefore such a philosophy cannot provide (generate) a principle (fundamental concept) applicable in an unquestionable manner for a large area of situations. That is why it turns out to be totally unacceptable (and useless) the idea to raise the entire UR philosophy to a rank of fundamental principle for QM.

Consequently, the precept **BP**₆ shows oneself as being nothing but an unjustified thesis. At the same time, from a true scientific perspective, it is outside of acceptable usages to put in practice an idea such is [10]: “*we use the name “uncertainty principle” simply because it is the most common one in the literature*”.

4 Which is really the true significance of UR?

Summing all the discussions incorporated within deficiencies **D**₁–**D**₁₂ one can notify the following evident remarks:

- There are profound deficiencies regarding all the basic elements and precepts of the conventional conceptions (CIUR doctrine and UR-QMS philosophy).
- In their essence the respective deficiencies are unavoidable and insurmountable within own framework of respective conceptions.
- Consequently the mentioned conceptions prove themselves as being undoubtedly in a failure situation which impose their abandonment.

The above argued abandonment of conventional conceptions points out very clearly the indubitable ending of the ex-

isting prevalent philosophy about UR. But a fair evaluation of such an ending requires an adequate epilogue regarding the future scientific status of the respective philosophy and of its constitutive and associate concepts.

The alluded epilogue demands firstly, detailed re-evaluations of the generative relations (1) and (2) from which have been expanded themselves the mentioned philosophy and concepts. The respective re-evaluations have to be done and argued by taking into account all the aspects noted previously within the texts of deficiencies D_1 – D_{12} . Doing so one arrives to the following observations:

- Relation (1) is nothing but an old-fashioned (and removable) empirical convention. It persists as a piece of historical reminiscence, destitute of any wonderful status/significance for actual and future physics.
- Relation (2) proves to be only an ordinary QM formula, of well-defined (but not universal) validity. In such a posture it describes in a simple manner the connections between fluctuation characteristics of two quantum observables.
- In fact the relations (1) and (2) have not any crucial significance, for QM concretely and less so for physics in general.
- Relations (1) and (2) or their “adjustments” have not any connection with genuine descriptions of QMS.
- Particularly the respective relations do not depict in any way the so called “observer effect” (i.e. perturbing influence of “experimenter” on the investigated system).

5 Considerations on quantum measurements

Besides the main discussions about the meaning of early relations (1) and (2), the conventional UR philosophy generated also many collateral debates on Quantum Measurements (QMS) (see [1–12, 86–88] and references). The respective debates, still active in writings of many scientists, promoted an appreciable diversity of viewpoints about conceptual significance and practical importance of QMS. But in the same context, were recorded observations like is the following one

- “Despite long efforts, no progress has been made ... for ... the understanding of quantum mechanics, in particular its measurement process and interpretation” [89].

Nevertheless, beyond the mentioned debates, the respective subject of QMS involves also a matter of real interest for physics. That matter regards the natural interest in developing adequate theoretical description(s) for QMS, which should to be proved through viable arguments and which have to become of suitable utility for scientific and technical activities.

The above signaled situation have motivated interest for both conventional and non-conventional approaches of QMS problem. A modest non-conventional approach was put in work progressively in our investigations over many years (see

[17–20, 47, 55, 90–94]). Here, as well as in all sections of present article, we try to gather, extend, systematize and improve the results of mentioned investigations in order to present argued viewpoints about the main aspects of QMS matter.

5.1 D_{13} : The incorrect association of QMS with UR

As a first main aspect of the so much debated QMS problem is fact that it has a theoretical essence. Namely, it is focused around the idea of developing a general theoretical model for describing measurements on quantum systems. The respective model should have some similarity (a bit of reference) with the one centered on Schrödinger equation within QM.

From the perspective of the such supposed similarity most of publications promoted or accepted the opinion that QMS have a basic essentiality for QM in itself. During the years were recorded even assertions like the following one:

- ‘the description of QMS is “probably the most important part of the theory (QM)”’ [5].

But note that both the mentioned opinion and assertion are grounded on the belief that, mainly, the claimed essentiality/importance of QMS for QM is given by relations (1) and (2) in terms of precepts BP_1 – BP_6 .

On the other hand, it is easy to see that the respective belief is invalidated by the arguments from the entire collection of deficiencies D_1 – D_{12} notified by us above in Section 3.

Now, besides the aforesaid notifications, for starting our non-conventional approach of QMS subject, we take into account the following remarks of J. S. Bell:

- “I agree with what you say about the uncertainty principle: it has to do with the uncertainty in predictions rather the accuracy of ‘measurement’. **I think in fact that the word ‘measurement’ has been so abused in quantum mechanics that it would good to avoid it altogether**” (see [83] and Appendix I below).
- “... The word (‘measurement’) has had such a damaging effect on the discussions that ... it should be banned altogether in quantum mechanics” [84].

A similar account we give also to the next remark:

- “the procedures of measurement (comparison with standards) has a part which cannot be described inside the branch of physics where it is used”. [95]

The just noted remarks consolidate for us the following key view:

- The significance of UR is an intrinsic question of QM while the description of QMS constitutes an adjacent but distinct subject comparatively with QM in itself.

As another reference element for starting our approach we agree the following observation:

- “it seems essential to the notion of measurement that it answers a question about the given situation existing before the measurement. Whether the measurement leaves the measured system unchanged or brings about a new and different state of that system is a second and independent question” [96].

In sense of above observation for a measured physical system the “situation existing before the measurement” regards the intrinsic properties of that system. The characteristics of the respective properties play a role of input data (information) for measuring actions. On the other hand for the same system, the “answer (i.e. result) of measurement” is accumulated in “output data (information)” that are provided by measuring process. Correspondingly the whole measurement can be considered as a transmission process for information (stochastic data), while the measuring device appears as a communication channel (viewed as in [97]).

So the whole image of a measurement can be depicted through the scheme

$$\left| \begin{array}{c} \text{input} \\ \text{data} \end{array} \right\rangle \Rightarrow \left[\begin{array}{c} \text{communication} \\ \text{channel} \end{array} \right] \Rightarrow \left[\begin{array}{c} \text{output} \\ \text{data} \end{array} \right]. \quad (24)$$

For giving concrete descriptions of the above scheme in cases of QMS (measurements on quantum systems) it should also take into view the next remark

- “To our best current knowledge the measurement process in quantum mechanics is non-deterministic” [89].

In such a view the mentioned input and output data as well the description of a QMS have to be presented by means of some non-deterministic (stochastic or random) entities. For a measured quantum system the totality of input data can be considered as being comprised in its specific (intrinsic) wave function Ψ_{in} , with known stochastic/probabilistic own significance. As regards the same system the output data should be represented by some quantities having also stochastic features. Formally, such quantities can be considered as being incorporated in an output wave function Ψ_{out} . Then the measuring process appear as communication channel which transposes the wave function from a Ψ_{in} reading into a Ψ_{out} image. So it can be suggested that, in case of a QMS, the scheme (24) can be represented through the following generic pattern:

$$\left| \begin{array}{c} \text{probabilistic} \\ \text{content of } \Psi_{in} \end{array} \right\rangle \Rightarrow [\widehat{SCC}] \Rightarrow \left[\begin{array}{c} \text{probabilistic} \\ \text{content of } \Psi_{out} \end{array} \right] \quad (25)$$

where \widehat{SCC} , depicts the “Stochastic Communication Channel” regarded as an “operator” which describe the measuring process.

The above suggested pattern regarding QMS can be particularized for various concrete situations by using QM terminology. Two such particularization will be detailed below in the Subsections 5.2 and 5.4.

5.2 On an observable with discrete spectrum

Let us refer to the case of a QMS for a single quantum observable A endowed with a non-degenerate discrete spectrum of eigenvalues $\{a_j\}_{j=1}^n$. The respective observable is described by the operator \hat{A} which satisfy the equations $\hat{A}\varphi_j = a_j \cdot \varphi_j$, where $\{\varphi_j\}_{j=1}^n$ signify the corresponding eigenfunctions.

If the set of eigenfunctions $\{\varphi_j\}_{j=1}^n$ is regarded as an orthonormal basis the wave functions Ψ_{in} and Ψ_{out} can be represented as follows

$$\begin{aligned} \Psi_{in} &= \sum_{j=1}^n \alpha_j \varphi_j, & \sum_{j=1}^n |\alpha_j|^2 &= 1, \\ \Psi_{out} &= \sum_{k=1}^n \beta_k \varphi_k, & \sum_{k=1}^n |\beta_k|^2 &= 1. \end{aligned} \quad (26)$$

Then the the pattern (25) appears as a transformation of the corresponding probabilities from *in*-readings $\{|\alpha_j|^2\}_{j=1}^n$ into *out*-images $\{|\beta_k|^2\}_{k=1}^n$. According to mathematics (probability and information theories) the mentioned transformation (i.e.the operator \widehat{SCC}) can be depicted by means of a doubly stochastic matrix M_{kj} ($k, j = 1, 2, \dots, n$), interpreted as in [98]. Such a depiction has the form

$$|\beta_k|^2 = \sum_{j=1}^n M_{kj} \cdot |\alpha_j|^2 \quad (27)$$

where the matrix M_{jk} satisfies the conditions

$$\sum_{k=1}^n M_{kj} = \sum_{j=1}^n M_{kj} = 1.$$

As above described a QMS appear as being ideal respectively non-ideal, accordingly as $M_{kj} = \delta_{kj}$ or $M_{kj} \neq \delta_{kj}$, where δ_{kj} denotes a Kronecker delta.

By using (26) and (27) for the η -expected values $\langle A \rangle_\eta = (\Psi_\eta, \hat{A}\Psi_\eta)$, ($\eta = in, out$), of observable A one obtains

$$\langle A \rangle_{in} = \sum_{j=1}^n a_j \cdot |\alpha_j|^2, \quad (28)$$

$$\langle A \rangle_{out} = \sum_{k=1}^n a_k \cdot |\beta_k|^2 = \sum_{k=1}^n \sum_{j=1}^n a_k \cdot M_{kj} \cdot |\alpha_j|^2.$$

In terms of above notations the error for the expected value of A is:

$$\mathcal{E}\{\langle A \rangle\} = \langle A \rangle_{out} - \langle A \rangle_{in} = \sum_{k=1}^n \sum_{j=1}^n a_k \cdot (M_{kj} - \delta_{kj}) \cdot |\alpha_j|^2 \quad (29)$$

where δ_{jk} signifies a Kronecker delta.

Because, mathematically, the observable A is a random variable it is characterized also by the standard deviations

$\Delta_\eta A$ ($\eta = in, out$), defined as follows

$$\begin{aligned} (\Delta_{in}A)^2 &= \langle (A - \langle A \rangle_{in})^2 \rangle_{in} \\ &= \sum_{j=1}^n a_j^2 \cdot |\alpha_j|^2 - \left(\sum_{j=1}^n a_j \cdot |\alpha_j|^2 \right)^2 \\ (\Delta_{out}A)^2 &= \langle (A - \langle A \rangle_{out})^2 \rangle_{out} \\ &= \sum_{k=1}^n \sum_{j=1}^n a_k^2 \cdot M_{kj} \cdot |\alpha_j|^2 \\ &\quad - \left(\sum_{k=1}^n \sum_{j=1}^n a_k \cdot M_{kj} |\alpha_j|^2 \right)^2 \end{aligned} \tag{30}$$

So for error $\mathcal{E}\{\Delta A\}$ of standard deviation regarding the observable A one finds

$$\begin{aligned} \mathcal{E}\{\Delta A\} &= \Delta_{out}A - \Delta_{in}A \\ &= \sqrt{\sum_{k=1}^n \sum_{j=1}^n a_k^2 \cdot M_{kj} \cdot |\alpha_j|^2 - \left(\sum_{k=1}^n \sum_{j=1}^n a_k \cdot M_{kj} |\alpha_j|^2 \right)^2} - \\ &\quad \sqrt{\sum_{j=1}^n a_j^2 \cdot |\alpha_j|^2 - \left(\sum_{j=1}^n a_j \cdot |\alpha_j|^2 \right)^2}. \end{aligned} \tag{31}$$

Now note the fact that, to some extent, the above presented model of a QMS description has general features. This because, excepting the conditions of being doubly stochastic, the measuring matrix M_{kj} can consists of arbitrary components. The mentioned generality/arbitrariness should be reduced when one refers to the relatively accurate measurements. Such a reduction can be modeled if the measuring matrix elements M_{kj} are taken of the forms

$$\begin{aligned} M_{kj} &= \delta_{kj} + \tau_{kj}, \\ |\tau_{kj}| &\ll 1, \quad \sum_{k=1}^n \tau_{kj} = \sum_{j=1}^n \tau_{kj} = 0, \end{aligned} \tag{32}$$

where δ_{kj} signifies the a Kronecker delta and τ_{kj} are real and dimensionless quantities of (very) small values.

When the matrix elements M_{jk} are approximated as in (32) the errors $\mathcal{E}\{\langle A \rangle\}$ and $\mathcal{E}\{\Delta A\}$ from (29) and (31) can be estimated through a direct calculation, respectively by means of the first order term in Taylor series. Then one finds

$$\begin{aligned} \mathcal{E}\{\langle A \rangle\} &= \sum_{k=1}^n \sum_{j=1}^n a_k \cdot \tau_{kj} \cdot |\alpha_j|^2, \\ \mathcal{E}\{\Delta A\} &\approx \sum_{k=1}^n \sum_{j=1}^n \left[\frac{\partial \mathcal{E}(\tau_{kj})}{\partial \tau_{kj}} \right]_{\tau_{kj}=0} \cdot \tau_{kj}, \end{aligned} \tag{33}$$

where $\mathcal{E}(\tau_{kj})$ signifies the standard-deviation error $\mathcal{E}\{\Delta A\}$ from (31) in which one uses the approximations (32).

Relations (33) show that within mentioned approximations the parameters τ_{jk} appear as significant indexes regarding the measuring accuracies. So the discussed measurement

can be regarded as ideal when $\tau_{kj} = 0$ for all k and j , respectively as non-ideal when $\tau_{kj} \neq 0$ at least for some values of k or j .

5.3 D_{14} : On the measuring scenarios with a unique sampling

As it was pointed out in Subsection 5.1, a QMS is essentially a non-deterministic process. Due to the mentioned essentiality, the “result” of such a process must be represented in terms of some stochastic (probabilistic) output data. But, surprisingly, in conventional publications [99–106] a QMS is regarded as a scenario (i.e. an imagined sequence of possible events) conceived as a single sampling (i.e. as a unique-deterministic selection from a set of random data). So regarded, a QMS gives as its result (outcome) a single value in which falls (collapses) the whole physical content of the measured observable. The referred falling scenarios are illustrated by two widely debated themes regarding the Wave Function Collapse (WFC) [99–103] respectively the Schrödinger’s Cat Thought Experiment (SCTE) [104–106]. Historically, both the respective themes have occurred in a direct connection with the establishing of basic precepts BP_1 – BP_6 of CIUR doctrine and UR–QMS philosophy. Therefore, by taking into account the deficiencies of precepts BP_1 – BP_6 , revealed above in Section 3, it is here the place to investigate also the possible deficiencies of the aforesaid scenarios.

Let us begin the announced investigation by referring to the WFC-measuring-scenario. The respective scenarios germinated from the hypothesis that, due to unavoidable measuring perturbations, all QMS cause specific collapses (falls, jumps) in states of the measured quantum systems. It can be presented succinctly in usual terms of QM as follows.

Consider a measuring investigation focused on the system and observable A discussed in the previous Subsection 5.2. For the respective system in WFC-scenario the “situation existing before measurement” is inscribed in its intrinsic wave function Ψ_{in} . The probabilistic content of Ψ_{in} play the role of input data (information) for investigation actions. But, attention, within the WFC-scenario, the measuring actions are imagined as providing as result an **unique deterministic outcome (udo)** namely a_k .

Note that a_k is one of the eigenvalues $\{a_j\}_{j=1}^n$ from the spectrum of A . The eigenvalues $\{a_j\}_{j=1}^n$ are defined through the relations $\hat{A}\varphi_j = a_j \cdot \varphi_j$ ($j = 1, 2, \dots, n$), where $\{\varphi_j\}_{j=1}^n$ denote the eigenfunctions of operator \hat{A} associated to the observable A . Then, in terms detailed previously in Subsection 5.2, the whole WFC-scenario can be illustrated through the following two schemes

$$\left\{ \{a_j\}_{j=1}^n \cup \{|\alpha_j|^2\}_{j=1}^n \right\} \Rightarrow [\widehat{udo}] \Rightarrow a_k, \tag{34}$$

$$\left| \Psi_{in} = \sum_{j=1}^n \alpha_j \cdot \varphi_j \right\rangle \Rightarrow [\widehat{udo}] \Rightarrow \varphi_k, \quad (35)$$

where \widehat{udo} symbolize an operator which describe the measuring actions in WFC-scenario.

On the one hand, firstly, the schema (34) regards the measurement of observable A . It show a falling of the respective observable from a whole spectrum of values $\{a_j\}_{j=1}^n$, having probabilities $\{|\alpha_j|^2\}_{j=1}^n$ in measured state, to a unique value a_k as result of the scenario. Secondly, on the other hand, the schema (35) refers to the evolution of the considered system from a state “existing before the measurement” (at the beginning of scenario) in an “after measurement” state (in the end of scenario).

Specify here the fact that conventional publications (see [99–103] and references) regard relation (35) as being the essential symbol of WFC. That is why the mentioned publications tried to done analytical representations of the respective relation considered as image of a dynamical physical process. For such representations were promoted various inventions, e.g. nonlinear extensions of Schrödinger equation or even appeals to new kinds of fundamental physical constants.

The above mentioned WFC-scenario regarding QMS can be admonished through the following remarks.

Firstly note that quantum observables are stochastic variables. Consequently a true measurement of such an observable should be regarded as being provided not by an **udo** (unique deterministic outcome) but by an adequate probabilistic set of such outcomes. The data given by the respective set are expected to provide relevant (and as complete as possible) information about the considered observables.

Secondly, the idea of describing QMS through an analytical representation of the WFC schema (35) proves oneself as being an extravagance without solid arguments or credible hypotheses. Some main aspects of the respective extravagance can be revealed by taking into account the stochastic similitude between quantum and thermal (macroscopic) random observables. Such a reveal we point out here as follows.

Let us refer to a macroscopic thermodynamic system described in terms of phenomenological theory of fluctuations (see below the Appendix D). For simplicity the system will be considered to be characterized by a single macroscopic thermodynamic observable \mathbb{A} . Mathematically the macroscopic fluctuations of \mathbb{A} are accounted by a real random variable \mathcal{A} and described by the probability density $W = W(\mathcal{A})$. Through the before specified terms can be pointed out the analogy between measuring acts regarding the stochastic observables of quantum and macroscopic nature. An **udo**, specific to WFC-scenario, for a quantum observable was discussed succinctly above in connection with the relations (34) and(35). A completely similar **udo** regarding a macroscopic observable \mathbb{A} can be depicted as follows. By means of an **udo** for the variable \mathcal{A} one obtains a unique value say \mathcal{A}_0 .

Then for \mathbb{A} the respective **udo** can be illustrated through the following relations

$$|\mathcal{A} \in (-\infty, +\infty)\rangle \Rightarrow [\widehat{udo}] \Rightarrow \mathcal{A}_0, \quad (36)$$

$$|W(\mathcal{A})\rangle \Rightarrow [\widehat{udo}] \Rightarrow \delta(\mathcal{A} - \mathcal{A}_0), \quad (37)$$

where $\delta(X)$ denotes the Dirac’s δ -function of X .

In principle, the aspects of quantum and macroscopic observables, depicted by (34) and (35) respectively (36) and (37) are completely similar. Therefore the discussions regarding the two kinds of **udo** should be similarly too. But in the macroscopic case the relation (37) is not considered at all as illustrating a dynamic process. Moreover within the corresponding macroscopic studies there is no interest for giving an analytical representation (through some evolution equations) regarding a scenario of type (37). This even if for the investigation of macroscopic observables one can use in principle a subjacent description given by classical statistical mechanics. Then, by virtue of above noted similarity, it can be said that the quantum scenario (35) should be not considered as a dynamic process. Consequently the QM studies have to be not concerned about the analytical representation (by some evolution equations) of an **udo** as the one illustrated by (35). Such regards about the scenario (35) are required, with all the more, as QM is not complemented (until today) by any subjacent theory of sub-quantum essence. Furthermore, for a true physical approach, the result of the respective **udo** must be gathered together with the answers of a significant statistical group of many other akin **udo**. The respective answers should allow to find adequate probabilistic estimators of the investigated quantum observable.

Regarding the problem of QMS description, in the category of falling scenarios, along with the WFC idea one finds also the famous problem of SCTE (Schrödinger’s Cat Thought Experiment). The respective problem, known also as Schrödinger’s cat paradox, has retained the attention of many debates over the decades (see [104–106] and references). The essential element in SCTE is represented by a single decay of an individual radioactive atom (which, through some macroscopic machinery, kills an initially living cat). But the individual lifetime of a single decaying atom is a stochastic (random) variable. That is why the mentioned killing decay is in fact a twin analogue of the above mentioned **udo** taken into account by the WFC-scenario. So, the above considerations reveal the notifiable fact that, for a true evaluation of a stochastic observable (such is the mentioned decay lifetime), is worthlessly to operate with an **udo** which gives an unique result of measurement. Accordingly, the SCTE problem appears as a twin analogue of the IWFC-scenario, i.e. as a fiction (figment) without any real scientific value.

The aforesaid fictional character of the SCTE can be pointed out once more by observation [93, 94] that it is possible to imagine a macroscopic thought-experiment completely

analogous with the SCTE. Within the respective macroscopic analogue, a cousin of Schrödinger's cat can be killed through launching a single macroscopic ballistic projectile. More specifically, the killing machinery is activated by an uncontrollable (unobservable) sensor located within the "circular error probable" (CEP) [109] of a ballistic projectile trajectory. The hitting point of the projectile is expected to arrive within CEP with the probability 50%. That is why the murderous action of a single launched projectile is just as much unpredictable as that of the unique radioactive atom within original SCTE. Therefore, the mentioned macroscopic analogy makes clear once more the fictional character of the SCTE.

According to the above-noted remarks, it should be regarded as worthless statements some assertions such as: "*The Schrödinger's cat thought experiment remains a defining touchstone for modern interpretations of quantum mechanics*" [106]. Note that such or similar assertions can be found in many popular publications or in the texts disseminated via the Internet (e.g. [110]).

Therefore SCTE problem as well as its similar WFC idea, discussed previously, prove themselves to be not real scientific topics but rather useless exercises (fictive scenarios), without any conceptual or practical significance.

5.4 About observables with continuous spectra

As it was noted in the beginning of this Section 5, for physics, development of suitable models for QMS description present a natural necessity. Above, in Subsection 5.2 of this article, it is detailed such a model regarding the measurement of an observable endowed with a discrete non-degenerate spectra. Here below we try to propose a measuring model with similar purpose (QMS description) but regarding observables having continuous spectra of values.

As in case with discrete spectrum for here regarded measuring situation we adopt the same generic pattern depicted in (25). The probabilistic content of wave functions Ψ_{in} and Ψ_{out} incorporate information (data) about the intrinsic state of the measured system respectively concerning the results provided by measurement. We will restrict our considerations to the measurements of orbital characteristics for a quantum spin-less micro-particle, supposed in a unidirectional motion along the x -axis. Note that the announced considerations can be easily extended for measurements regarding systems with spatial orbital motions. Then the wave functions Ψ_η ($\eta = in, out$) will be taken of the form $\Psi_\eta = \Psi_\eta(x)$ (note that here we omit to specify the time t as visible variable because the considered state of system refers to a given ante-measurement instant).

Note now the fact that according QM rules the wave functions Ψ_η have only significance of probability amplitudes but not a direct probability meaning. Therefore, in the case of interest here, the picture (25) of QMS should be detailed not in terms of wave functions Ψ_η , but by means of some entities

with direct probabilistic meanings. This especially because the real measuring devices report the occurrence of some random values for investigated observables. In usual terms of QM entities with direct probabilistic significance are carriers of stochasticity: probability densities ρ_η and probability currents j_η ($\eta = in, out$). Let us write the wave functions Ψ_η as $\Psi_\eta(x) = |\Psi_\eta(x)| \cdot \exp\{i\Phi_\eta(x)\}$. Then, for a micro-particle with mass m considered as measured system, the alluded ρ_η and j_η are given by relations:

$$\begin{aligned}\rho_\eta &= \rho_\eta(x) = |\Psi_\eta(x)|^2, \\ j_\eta &= j_\eta(x) = \frac{\hbar}{m} |\Psi_\eta(x)|^2 \cdot \nabla_x \Phi_\eta(x),\end{aligned}\quad (38)$$

where $\nabla_x = \frac{\partial}{\partial x}$.

Now it must to specify that ρ_η and j_η refer to the positional and the motional kinds of probabilities respectively. Experimentally the two kinds can be regarded as measurable by distinct devices and procedures. The situation is similar with that of electricity studies where the aspects regarding position and mobility of electrical charges are evaluated through completely different devices and procedures. Due to the aforesaid specifications it results that in fact the generic pattern depicted in (25) has to be amended as follows

$$|\rho_{in}(x) \cup j_{in}(x)\rangle \Rightarrow [\widehat{SCC}] \Rightarrow [\rho_{out}(x) \cup j_{out}(x)]. \quad (39)$$

Mathematical considerations about the relations (25) and (E.1), (early referred also in [107]) can be applied by similarity for the pattern (39). So the respective pattern (i.e. the operator \widehat{SCC}) can be represented through the next two transformations:

$$\begin{aligned}\rho_{out}(x) &= \int_{-\infty}^{+\infty} \Gamma(x, x') \cdot \rho_{in}(x') \cdot dx', \\ j_{out}(x) &= \int_{-\infty}^{+\infty} \Lambda(x, x') \cdot j_{in}(x') \cdot dx' .\end{aligned}\quad (40)$$

Here $\Gamma(x, x')$ and $\Lambda(x, x')$ represent the corresponding doubly stochastic kernels (in sense defined in [108]). This means that the kernels $\mathfrak{K} = \{\Gamma, \Lambda\}$ satisfy the following relations $\int_{-\infty}^{+\infty} \mathfrak{K}(x, x') dx = \int_{-\infty}^{+\infty} \mathfrak{K}(x, x') dx' = 1$. The mentioned kernels incorporate some extra-QM elements regarding the characteristics of measuring devices and procedures. Such elements do not belong to the usual QM framework which refers to the intrinsic (own) characteristics of the measured micro-particle (system).

Through the above considerations can be evaluated the effects induced by QMS. The respective effects regards the probabilistic estimators for orbital observables A_j of considered quantum system. Such observables are described by the

operators \hat{A}_j ($j = 1, 2, \dots, n$). As in case of classical measuring model (see the Appendix E), without any loss of generality, here one can suppose that the quantum observables have identical spectra of values in both *in*-and *out*-situations. In terms of QM the mentioned supposition means that the operators \hat{A}_j have the same mathematical expressions in both *in*-and *out*-readings, i.e. that the respective expressions remain invariant under the transformations which describe QMS. In the here discussed case of a system with rectilinear orbital motion the mentioned expressions depend on x and ∇_x .

So one can say that in the situations associated with the wave functions $\Psi_\eta = \Psi_\eta(x)$ ($\eta = in, out$) the mentioned quantum observables A_j , can be characterized by the following lower order estimators (or numerical parameters): mean values $\langle A_j \rangle_\eta$, correlations $C_\eta(A_j, A_k)$ and standard deviations $\Delta_\eta A_j$. We use the common notation (f, g) for scalar product of functions f and g , i.e. $(f, g) = \int_{-\infty}^{+\infty} f^*(x) \cdot g(x) \cdot dx$. Then the mentioned estimators are defined by the relations

$$\begin{aligned} \langle A_j \rangle_\eta &= (\Psi_\eta, \hat{A}_j \Psi_\eta), & \delta_\eta \hat{A}_j &= \hat{A}_j - \langle A_j \rangle_\eta, \\ C_\eta(A_j, A_k) &= (\delta_\eta \hat{A}_j \Psi_\eta, \delta_\eta \hat{A}_k \Psi_\eta), & & \\ \Delta_\eta A_j &= \sqrt{C_\eta(A_j, A_j)}. \end{aligned} \tag{41}$$

Note here the fact that, on the one hand, the *in*-version of discussions the estimators (41) are calculated by means of the wave function Ψ_{in} . The respective function is supposed as being known from the considerations about the intrinsic properties of the investigated system (e.g. by solving the corresponding Schrödinger equation).

On the other hand, apparently, the evaluation of estimators (41) in $\eta = out$ -version requires to operate with the wave function Ψ_{out} . But the respective appearance can be surpassed [20] through operations which use the probability density ρ_{out} and current j_{out} . So if an operator \hat{A}_j does not depend on $\nabla_x = \frac{\partial}{\partial x}$, i.e. $\hat{A}_j = \hat{A}_j(x)$, in evaluating the scalar products from (41) can be used the evident equality $\Psi_{out}^* \hat{A}_j \Psi_{out} = \hat{A}_j \cdot \rho_{out}$. Additionally, when \hat{A}_j depends on $\nabla_x = \frac{\partial}{\partial x}$, i.e. $\hat{A}_j = \hat{A}_j(\nabla_x)$, in the same products the expressions of the type $\hat{A}_j(\nabla_x) \Psi_{out}(x)$ can be converted in terms of $\rho_{out}(x)$ and $j_{out}(x)$. Namely from (38) one finds directly:

$$\begin{aligned} \nabla_x |\Psi_{out}(x)| &= \nabla_x \sqrt{\rho_{out}(x)}, \\ \nabla_x \Phi_{out}(x) &= \frac{m j_{out}(x)}{\hbar \rho_{out}(x)}. \end{aligned} \tag{42}$$

By a single or repeated application of these formulas, any expression of type $\hat{A}_j(\nabla_x) \Psi_{out}(x)$ can be transcribed in terms of ρ_{out} and j_{out} .

The aforesaid discussion should be supplemented by specifying some indicators able to characterize the errors (uncertainties) of considered QMS. For the above quoted observ-

ables A_j such indicators are the following ones:

$$\left. \begin{aligned} \mathcal{E}\{\langle A_j \rangle\} &= \langle A_j \rangle_{out} - \langle A_j \rangle_{in} \\ \mathcal{E}\{C(A_j, A_k)\} &= C_{out}(A_j, A_k) - C_{in}(A_j, A_k) \\ \mathcal{E}\{\Delta A_j\} &= \Delta_{out} A_j - \Delta_{in} A_j \end{aligned} \right\} \tag{43}$$

The above presented model regarding the description of QMS for observables with continuous spectra is illustrated on a simple example in the Appendix G below.

6 Some concluding remarks

The present paper was motivated by the existence of many unclearnesses (unfinished controversies and unelucidated questions) about the prevalent UR–QMS philosophy. It was built as a survey on deficiencies of respective philosophy. So were re-evaluated the main ideas claimed within the mentioned philosophy. The basic results of the respective re-evaluations can be summarized through the following **Concluding Remarks (CR)**:

- **CR₁**: Firstly, through multiple arguments, we have proved the observation that the UR (1) and (2) have not any essential significance for physics. Namely the respective UR are revealed as being either old-fashioned, short-lived (and removable) conventions (in empirical, thought-experimental justification) or simple (and limited) mathematical formulas (in theoretical vision). But such an observation comes to advocate and consolidate the Dirac’s intuitive prediction [23]: “*I think one can make a safe guess that uncertainty relations in their present form will not survive in the physics of future*”. Note that the respective prediction was founded not on some considerations about the UR essence but on an intuition about the future role in physics of Planck’s constant \hbar . Dirac predicted that \hbar will become a derived (secondary) quantity while c and e will remain as fundamental constants ($c =$ speed of light and $e =$ elementary electric charge). ■

- **CR₂**: A significant idea that emerges from previous discussions is the one that neither UR (1) and (2) nor various “generalizations” of them, have not any connection with genuine descriptions of Quantum measurements (QMS). All the respective descriptions should be considered as a distinct (and additional) subject which must be investigated separately but somewhat in association with QM. Examples of such description are presented briefly, in Subsection, 5.2 and 5.4, for observables having discrete respectively continuous spectra. ■

- **CR₃**: Note that, in all of their aspects, the discussions from Subsection 5.2 and 5.4 have a theoretical essence. This means that, the entities like wave function Ψ_{in} as well as the measuring indicators M_{jk} , $\Gamma(x, x')$ and $\Lambda(x, x')$, are nothing but abstract concepts which enable elaboration of theoretical models regarding the descriptions of QMS. On the one hand Ψ_{in} refers to the intrinsic data about the studied system. It is

evaluated by means of some known theoretic procedures (e.g. by means of the corresponding Schrödinger equation). On the other hand the indicators M_{jk} , $\Gamma(x, x')$ and $\Lambda(x, x')$ are introduced as theoretical entities for modeling the characteristics of the considered measuring devices/processes. ■

- **CR₄**: Correlated with the previous **CR₂** and **CR₃** it must be specified that, in relation with QMS, the inventions of Wave Function Collapse (WFC) and Schrödinger's Cat Thought Experiment (SCTE) are nothing but unnatural falling scenarios. Consequently, as we have argued above in Subsections 5.3, both idea of WFC and SCTE problem prove themselves as being not real scientific subjects but rather unnecessary figments. ■

- **CR₅**: It is interesting to note here the fact that the history of quantum mechanics was abounded by an impressive number of publications related to UR–QMS philosophy. So, for the years between 1935 and 1978, as regards EPR (Einstein-Podolsky-Rosen) paradox, associated [112] with the situation of non-commuting observables, some authors [113] noted that “ $\geq 10^6$ papers have been written” — i.e. ≥ 63 papers per day (!?). Also the same publishing abundance about QM matters (including UR–QMS philosophy) motivates remarks such are the following ones: “*A theory whose formalism can be written down on a napkin whilst attempts to interpret it fill entire libraries. A theory that has seen astonishing experimental conformation yet leaves us increasingly perplexed the more we think about it. How can we know so well how to apply this theory but disagree so vehemently about what it is telling us?*” [114]. Probably that, in some future, the alluded abundance will be investigated from historic and sociological perspectives.

- **CR₆**: Over the years original UR (1) and (2) were supplemented with many kinds of “generalizations” (see [115–120] and references). Until today, the respective “generalizations” appear as being de facto only extrapolation mathematical “constructs” (often of impressive inventiveness). As a rule, they are not pointed out as having significance for some concrete physical questions (of conceptual or experimental relevance). But the existence of such significance is absolutely necessary in order to associate the mentioned “generalizations” with matters of certain importance for physics. In the light of the discussions from the present paper one can say that the sole physical significance of some from the referred “generalizations” seems to be their meaning as quantitative indicators of fluctuations (i.e. of stochasticity). But from a practical perspective among the respective indicators of practical usage are only the ones of relative lower order. Therefore, for tangible interests of physics, all the discussed “generalizations” seem to be rather excessive pieces. They remain only as interesting mathematical “constructs”, which ignore the desideratum: “*Entities are not to be multiplied beyond necessity*”. ■

- **CR₇**: In discussions and revaluations proposed in this article, we have referred only to the prevalent philosophy of

UR and QMS regarding primarily the foundations and interpretation of QM. But, as it is known, the mentioned philosophy has been extrapolated in other “extra muros” domains, outside of QM. As aforesaid domains can be quoted the following ones: mathematical computations, biology and medical sciences, economy and finance, human behavior, social sciences and even politics. A relevant bibliography regarding the mentioned extramural extrapolations can be accessed easy via internet from Google sites. Note that our above reevaluations of UR–QMS philosophy do not contain analyzes referring to the mentioned extrapolations. Such analyzes remain as task for scientists working in the respective domains. Here we want to point out only one noticeable aspect that differentiates the extramural UR from the primary ones. On the one hand, according to their origin, the primary UR from QM are strongly associated with a cardinal marker represented by the Planck constant \hbar . On the other hand, as far as we know, for extramural extrapolations of UR, the existence of similar markers, represented by cardinal indicators of the corresponding scientific domains, were not reported. ■

- **CR₈**: In their essence, the above argued revaluations of UR and QMS, do not disturb in any way the basic framework of usual QM. This means that QM keeps its known specific elements: concepts (wave functions, operators) with their significances (of stochastic essence), principles and theoretical models (Schrödinger equation), computing rules (exact or approximate) and investigate systems (atoms, molecules, mesoscopic structures). Note here the observation that, for nowadays existing quantum debates, the above revaluations of UR–QMS, offer a few arguments for lucrative parsimony in approaches of matters. The unlucrative aspects of those debates have to be reconsidered too, probably in more or less speculative visions. We recall here that the basic framework of QM can be deduced [121] from direct physical considerations, without appeals to ambiguous discussions about UR or QMS. The alluded considerations start from real physical facts (particle-wave duality of atomic size systems). Subsequently they use the continuity equations for genuine probability density and current. After that one obtains the whole framework of QM (i.e. the Schrödinger equation, expressions of operators as descriptors of quantum observables and all the practical rules of QM regarded as a theoretical model for the corresponding investigated systems).

In the mentioned perspective, we dare to believe that, to some extent, the revaluations of UR and QMS promoted by us can give modest support for genuine reconsiderations regarding the interpretation and foundations of QM. ■

Appendices

A: A brief compendium of some QM elements

Here we remind briefly some significant elements, selected from the usual theoretical framework [5, 29, 30] of Quantum Mechanics (QM). In this appendix we use Traditional Nota-

tions (TN), taken over from mathematical algebra developed long before QM appeared. Few specifications about the more recent Dirac's bracket formalism are given in Appendix B.

So, in terms of TN, we consider a QM micro-particle whose state (of orbital nature) is described by the wave function Ψ . Two observables A_j ($j = 1, 2$) of the respective particle will be described by the operators \hat{A}_j . The notation (f, g) will be used for the scalar (inner) product of the functions f and g . Correspondingly, the quantities $\langle \hat{A}_j \rangle_\Psi = (\Psi, \hat{A}_j \Psi)$ and $\delta_\Psi \hat{A}_j = \hat{A} - \langle \hat{A}_j \rangle_\Psi$ will depict the mean (expected) value respectively the deviation-operator of the observable A_j regarded as a random variable. Then, by denoting two observables with $A_1 = A$ and $A_2 = B$, one can be written the following formula:

$$\begin{aligned} (\delta_\Psi \hat{A} \Psi, \delta_\Psi \hat{A} \Psi) \cdot (\delta_\Psi \hat{B} \Psi, \delta_\Psi \hat{B} \Psi) &\geq \\ &\geq \left| (\delta_\Psi \hat{A} \Psi, \delta_\Psi \hat{B} \Psi) \right|^2 \end{aligned} \quad (A.1)$$

which is nothing but a relation of Cauchy-Schwarz type from mathematics.

For an observable A_j considered as a random variable, in a mathematical sense, the quantity $\Delta_\Psi A_j = (\delta_\Psi \hat{A}_j \Psi, \delta_\Psi \hat{A}_j \Psi)^{1/2}$ signifies its standard deviation. From (A.1) it results directly that the standard deviations $\Delta_\Psi A$ and $\Delta_\Psi B$ of the mentioned observables satisfy the formula

$$\Delta_\Psi A \cdot \Delta_\Psi B \geq \left| (\delta_\Psi \hat{A} \Psi, \delta_\Psi \hat{B} \Psi) \right|. \quad (A.2)$$

This last formula, with quantities $\Delta_\Psi A$ and $\Delta_\Psi B$ regarded together, play an influential role in QM debates within UR and QMS philosophy. That is why the relation (A.2) can be called *Cauchy-Schwarz Quantum Formula* (CSQF). Note that formulas (A.1) and (A.2) are always valid, i.e. for all observables, particles and states. Therefore they must be considered as primary QM formulas.

For the discussions regarding the UR-QMS philosophy it is helpful to present the particular versions of formula (A.1) in the cases when the operators $\hat{A} = \hat{A}_1$ and $\hat{B} = \hat{A}_2$ satisfy the conditions

$$\text{iff} : (\hat{A}_j \Psi, \hat{A}_k \Psi) = (\Psi, \hat{A}_j \hat{A}_k \Psi), \quad (j, k = 1, 2) \quad (A.3)$$

(where iff \equiv if and only if). In the alluded cases it is true the next formula

$$\begin{aligned} (\delta_\Psi \hat{A} \Psi, \delta_\Psi \hat{B} \Psi) &= \frac{1}{2} (\Psi, \{ \delta_\Psi \hat{A}, \delta_\Psi \hat{B} \} \Psi) \\ &\quad - \frac{i}{2} (\Psi, i [\hat{A}, \hat{B}] \Psi). \end{aligned} \quad (A.4)$$

Here $\{ \hat{A}, \hat{B} \} = \hat{A} \hat{B} + \hat{B} \hat{A}$ and $[\hat{A}, \hat{B}] = \hat{A} \hat{B} - \hat{B} \hat{A}$ signify the anti-commutator respectively commutator of the operators \hat{A} and \hat{B} . Now note the fact that the two terms from the right hand side of (A.4) are purely real and strictly imaginary quantities respectively. Therefore in the mentioned cases from (A.2)

follows directly the enlarged inequality

$$\begin{aligned} (\Delta_\Psi A)^2 \cdot (\Delta_\Psi B)^2 &\geq \frac{1}{4} \left| \langle \{ \delta_\Psi \hat{A}, \delta_\Psi \hat{B} \} \rangle_\Psi \right|^2 \\ &\quad + \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle_\Psi \right|^2. \end{aligned} \quad (A.5)$$

Sometimes this relation is referred to as the Schrödinger inequality. It imply subsequently the next two truncated inequalities

$$\Delta_\Psi A \cdot \Delta_\Psi B \geq \frac{1}{2} \left| \langle \{ \delta_\Psi \hat{A}, \delta_\Psi \hat{B} \} \rangle_\Psi \right|, \quad (A.6)$$

$$\Delta_\Psi A \cdot \Delta_\Psi B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle_\Psi \right|. \quad (A.7)$$

One observes that (A.7) is nothing more than the conventional Robertson-Schrödinger relation (2), commonly quoted in the literature of CIUR doctrine and UR-QMS philosophy. Note that in the respective literature besides the relation (2)/(A.7) sometimes the formula (A.5) is also mentioned. But, as a fact, the respective mention is not accompanied with the important specification that formula (A.5) is valid iff (if and only if) the condition (A.3) is fulfilled.

In the end of this appendix we note the cases of more than two observables, i.e. for a set A_j ($j = 1, 2, \dots, n; n \geq 3$), when the quantities $\alpha_{jk} = (\delta_\Psi \hat{A}_j \Psi, \delta_\Psi \hat{A}_k \Psi)$ constitute the components of a positive semi definite matrix. In such cases, similarly with (A.1), are true the formulas

$$\det \left[(\delta_\Psi \hat{A}_j \Psi, \delta_\Psi \hat{A}_k \Psi) \right] \geq 0; \quad (j, k = 1, 2, \dots, n) \quad (A.8)$$

where $\det [\alpha_{jk}]$ is the determinant whose components are the quantities α_{jk} .

Note that within dominant publications promoted by the UR-QMS philosophy the interpretation of many-observable relations (A.8) is frequently omitted. The omission is due most probably to the fact that the idea of referring to simultaneous measurements for more than two observables is not supported convincingly by the current practice of experimental physics.

Addendum

Sometimes, in QM practice, a wave function Ψ is represented as a superposition of the form

$$\Psi = \sum_n \alpha_n \cdot \varphi_n, \quad \sum_n |\alpha_n|^2 = 1, \quad (A.9)$$

where $\{ \varphi_n \}$ denote a complete set of orthonormal basic functions for which $(\varphi_n, \varphi_m) = \delta_{nm} =$ a Kronecker delta.

Then, in a state described by Ψ , the mean value of an observable A is written as

$$\langle A \rangle_\Psi = \sum_{n,m} \alpha_n^* \cdot A_{nm} \cdot \alpha_m, \quad A_{nm} = (\varphi_n, \hat{A} \varphi_m), \quad (A.10)$$

with A_{nm} indicating the matrix elements of operator \hat{A} in representation given by $\{\varphi_n\}$.

When $\{\varphi_n\}$ are eigenfunctions of \hat{A} the following formulas can be written

$$\hat{A} \varphi_n = a_n \cdot \varphi_n, \quad \langle A \rangle_\Psi = \sum_n |\alpha_n|^2 \cdot a_n, \quad (\text{A.11})$$

where a_n signify the eigenvalue of \hat{A} in respect with the eigenfunction φ_n .

Note that the notations and formulas reminded in this Addendum can be used in connection with all quantities discussed above in present Appendix.

B: On the omission of conditions (A.3) within current literature

The mentioned omission is encountered in many generally agreed publications on QM (especially in textbooks, e.g. in [29]). It appears when the conventional Robertson-Schrödinger relation (A.7) is established by starting from the correct formula

$$\|((\delta_\Psi \hat{A} + i\lambda \delta_\Psi \hat{B}) \Psi)\| \geq 0 \quad (\text{B.1})$$

for the norm $\|f\|$ of function $f = (\delta_\Psi \hat{A} + i\lambda \delta_\Psi \hat{B}) \Psi$. In (B.1) are used the notations presented in the previous Appendix A and λ denote a real and arbitrary parameter. In order to go on from this last formula to the relation (A.5), it is presumed the equality

$$\begin{aligned} & ((\delta_\Psi \hat{A} + i\lambda \delta_\Psi \hat{B}) \Psi, (\delta_\Psi \hat{A} + i\lambda \delta_\Psi \hat{B}) \Psi) = \\ & (\Psi, (\delta_\Psi \hat{A})^2 \Psi) + \lambda^2 (\Psi, (\delta_\Psi \hat{B})^2 \Psi) \\ & - i\lambda (\Psi, [\hat{A}, \hat{B}] \Psi). \end{aligned} \quad (\text{B.2})$$

Then, due to the fact that λ is a real and arbitrary quantity, from (B.1) it results the relation

$$\left\langle (\delta_\Psi \hat{A})^2 \right\rangle_\Psi \cdot \left\langle (\delta_\Psi \hat{B})^2 \right\rangle_\Psi \geq \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle_\Psi \right|^2. \quad (\text{B.3})$$

In terms of notations from Appendix A this last relation gives directly the formula

$$\Delta_\Psi A \cdot \Delta_\Psi B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle_\Psi \right| \quad (\text{B.4})$$

which is nothing but the relation (A.7) from the previous Appendix.

Observation: Note here the next two aspects: (i) Introduction of (B.4) demands with necessity the existence of equality (B.2), (ii) The respective equality is true only when the operators \hat{A} and \hat{B} satisfy the conditions (A.3). The noted aspects must be signalized as omissions of the current literature.

Another context in which appears the omission of conditions (A.3) is connected with the “braket notation” frequently used in QM literature. Within the respective notation, known

also as *Dirac’s Notation* (DN), the scalar (inner) product of two functions f and g is depicted as $\langle f | g \rangle$ (see [29–31]). Of course DN was used in many texts regarding UR–QMS philosophy. But it must be pointed out the fact that in those texts the condition (A.3), justified in the previous Appendix, is totally omitted and its implications are not analyzed at all. It is easy to notice that such an omission is due to the fact that, within the DN, both terms (from left-hand and right-hand sides) of the condition (A.3) have the same transcription, namely:

$$\begin{aligned} & (\hat{A}_j \Psi, \hat{A}_k \Psi) = \langle \Psi | \hat{A}_j \hat{A}_k | \Psi \rangle \\ & \text{and} \\ & (\Psi, \hat{A}_j \hat{A}_k \Psi) = \langle \Psi | \hat{A}_j \hat{A}_k | \Psi \rangle. \end{aligned} \quad (\text{B.5})$$

Obviously, such transcriptions create confusion and obstruct the just consideration of the condition (A.3) for cases where it is absolutely necessary in debates about UR–QMS philosophy. In order to avoid the above mentioned confusion in [32] we suggested that DN to be replaced by an Improved Dirac Notation (IDN). For such an IDN we proposed, that within scalar product of two functions f and g , to insert additionally the symbol “•” so that the respective product to be depicted as $\langle f | \bullet | g \rangle$. In such a way it becomes directly visible the separation of the entities implied in that product. Then, inside of IDN, the two terms from (A.3) are transcribed as

$$\begin{aligned} & (\hat{A}_j \Psi, \hat{A}_k \Psi) = \langle \Psi | \hat{A}_j \bullet \hat{A}_k | \Psi \rangle \\ & \text{and} \\ & (\Psi, \hat{A}_j \hat{A}_k \Psi) = \langle \Psi | \bullet \hat{A}_j \hat{A}_k | \Psi \rangle \end{aligned} \quad (\text{B.6})$$

Now one observes that in terms of IDN the condition (A.3) appears in the form

$$\text{iff} \quad \langle \Psi | \hat{A}_j \bullet \hat{A}_k | \Psi \rangle = \langle \Psi | \bullet \hat{A}_j \hat{A}_k | \Psi \rangle \quad (\text{B.7})$$

which no longer generates confusions in discussions about UR–QMS philosophy.

C: Classical “uncertainty relations” in Fourier analysis

In classical mathematical harmonic analysis it is known a relation (often named theorem) which, in terms of here used notations, is similar with the quantum UR depicted by relation (2). Through current mathematical representations the respective relation can be introduced as follows.

Let be a pair of variables x and ξ , with domains $x \in (-\infty, +\infty)$ and $\xi \in (-\infty, +\infty)$, regarded as arguments of a function $f(x)$ respectively of its Fourier transform

$$\tilde{f}(\xi) = \int_{-\infty}^{+\infty} \exp(-2i\pi\xi \cdot x) \cdot f(x) \cdot dx. \quad (\text{C.1})$$

If the norm $\|f\|$ of $f(x)$ has the property $\|f\| = 1$, both $|f(x)|^2$ and $|\tilde{f}(\xi)|^2$ are probability density functions for x and ξ regarded as real random (stochastic) variables. The variances

of such variables, evaluated through the corresponding probabilities, can be noted as $\langle(x - \langle x \rangle)^2\rangle$ and $\langle(\xi - \langle \xi \rangle)^2\rangle$. The respective variances express the effective widths of functions $f(x)$ and $\tilde{f}(\xi)$. Then [66] the aforesaid relation/theorem is given by the formula

$$\langle(x - \langle x \rangle)^2\rangle \cdot \langle(\xi - \langle \xi \rangle)^2\rangle \geq \frac{1}{16\pi^2}. \tag{C.2}$$

In mathematics this formula express the fact that: “A nonzero function and its Fourier transform cannot both be sharply localized” [66].

Often formula (C.2) is transcribed in a equivalent variant as follows

$$\Delta x \cdot \Delta \xi \geq \frac{1}{4\pi} \tag{C.3}$$

where Δx and $\Delta \xi$ denote the corresponding standard deviations of x and ξ , defined through conventions like $\Delta x = \sqrt{\langle(x - \langle x \rangle)^2\rangle}$. In non-quantum physics a version of relation (C.3) appears in studies of classical signals (waves of acoustic or electromagnetic nature) where $x = t = \text{time}$ and $\xi = \nu = \text{frequency}$. The respective version is written as

$$\Delta t \cdot \Delta \nu \geq \frac{1}{4\pi} \tag{C.4}$$

and it is known [67] as “Gabor’s uncertainty relation”. This last relation (C.4) means the fact that, for a classical signal (regarded as a wave packet), the product of the “uncertainties” (“irresolutions”) Δt and $\Delta \nu$ in time and frequency domains cannot be smaller than a specific constant.

Formally the classical relation (C.3) can be transposed to the case of “quantum wave packets” often discussed in introductory/intuitive texts about QM. Such a transposition focuses on the pairs of conjugated observables $q-p$ (coordinate-momentum) respectively $t-E$ (time-energy). The corresponding transpositions can be obtained by setting in (C.4) the substitutions $x = q$ and $\xi = p(2\pi\hbar)^{-1}$ respectively $x = t$ and $\xi = E(2\pi\hbar)^{-1}$. The substitutions of variable ξ are nothing but the so called duality relations (regarding the wave-particle connections). By means of the mentioned substitutions from (C.4) one finds the following two relations

$$\Delta q \cdot \Delta p \geq \frac{\hbar}{2} \quad \text{respectively} \quad \Delta t \cdot \Delta E \geq \frac{\hbar}{2}. \tag{C.5}$$

These last formulas are similar with the conventional UR (2) for the pairs of observables $q-p$ respectively $t-E$. Note that the mentioned similarity is admissible iff (if and only if) one accepts the conventions $|\langle[\hat{q}, \hat{p}]\rangle_\Psi| = \hbar$ and $|\langle[\hat{t}, \hat{E}]\rangle_\Psi| = \hbar$. But attention, the last convention has no more than a “metaphoric” value. This because in usual QM framework the time t is a deterministic but not random (stochastic) variable and, genuinely, for the respective framework a *time operator* \hat{t} is nothing but a senseless and fictitious concept (see also the discussions from the deficiency \mathbf{D}_8).

Note that the classical relation (C.3) can be transposed also in another quantum formula regarding the ground state of a Quantum Torsion Pendulum (QTP) (see Subsection 3.6.2). For respective transposition in (C.3) it should to take $f(x) = \Psi(\varphi)$, $x = \varphi$ and $\xi = L_z \cdot (2\pi\hbar)^{-1}$. So one obtains the formula

$$\Delta \varphi \cdot \Delta L_z \geq \frac{\hbar}{2} \tag{C.6}$$

which is nothing but the lowest level version of the last of formulas (13)

Addendum

It is worth to mention here the fact that, in the Fourier analysis, the x -unlimited relations (C.3) and (C.4) have correspondent formulas in x -limited cases (when the variable x has a finite domain of existence). The respective fact can be evidenced as follows.

Let be $x \in [0, b]$, with b a finite quantity and function $f(x)$ having the property $f(0) = f(b-0) := \lim_{x \rightarrow b-0} f(x)$. Then the quantities

$$c_n = \frac{1}{\sqrt{b}} \int_0^b \exp(-ik_n x) \cdot f(x) \cdot dx \tag{C.7}$$

represent the Fourier coefficients of $f(x)$, with $k_n = n \cdot \frac{2\pi}{b}$ and n denoting integers i.e. $n \in \mathbb{Z}$.

Moreover if the measure $|f(x)|^2 dx$ denotes the infinitesimal probability for $x \in (x, x+dx)$ the quantity $|c_n|^2$ signify the discrete probability associated to the value k_n . Then for functions $A = A(x)$ and $B = B(k_n)$, depending on x respectively on k_n , the mean (expected) values $\langle A \rangle$ and $\langle B \rangle$ are written as follows

$$\langle A \rangle = \int_0^b A(x) \cdot |f(x)|^2 dx, \tag{C.8}$$

$$\langle B \rangle = \sum_n B(k_n) \cdot |c_n|^2.$$

As the most used such mean (expected) values can be quoted the following ones: first order moments $\langle x \rangle$ and $\langle k_n \rangle = \langle k \rangle$, variances $\langle(x - \langle x \rangle)^2\rangle$ and $\langle(k_n - \langle k \rangle)^2\rangle$ respectively standard deviations $\Delta x = \sqrt{\langle(x - \langle x \rangle)^2\rangle}$ and $\Delta k = \sqrt{\langle(k_n - \langle k \rangle)^2\rangle}$.

In order to find the announced x -limited correspondents of x -unlimited relations (C.3) and (C.4) we take into account the following obvious formula

$$\int_0^b \left| \lambda(x - \langle x \rangle) \cdot f(x) + \left(\frac{d}{dx} - i \langle k \rangle \right) \cdot f(x) \right|^2 \cdot dx \geq 0 \tag{C.9}$$

where λ is a real, finite and arbitrary parameter. By using the above noted probabilistic properties of function $f(x)$ and coefficients c_n from (C.9) one obtains the relation

$$\lambda^2 \langle(x - \langle x \rangle)^2\rangle + \lambda (b |f(0)|^2 - 1) + \langle(k - \langle k \rangle)^2\rangle \geq 0. \tag{C.10}$$

Due to the mentioned characteristics of λ , from this last relation one finds the next formulas for variances of x and k_n

$$\langle (x - \langle x \rangle)^2 \rangle \cdot \langle (k_n - \langle k \rangle)^2 \rangle \geq \frac{1}{4} (b |f(0)|^2 - 1)^2 \quad (C.11)$$

respectively for standard deviations of x and k_n

$$\Delta x \cdot \Delta k \geq \frac{1}{2} \left| \left(b |f(0)|^2 - 1 \right) \right|. \quad (C.12)$$

The formulas (C.11) and (C.12) are x -limited analogues of the x -unlimited relations (C.2) and (C.3).

In the end we note that formula (C.12) is applicable in cases of wave functions (4) regarding non-degenerate circular rotations. For such cases the application of (C.12) is obtained through the following substitutions: $x \rightarrow \varphi$, $b \rightarrow 2\pi$, $f(x) \rightarrow \Psi(\varphi)$ and $k_n \rightarrow \frac{L_z}{\hbar}$. So from (C.12) it results

$$\Delta \varphi \cdot \Delta L_z \geq \frac{\hbar}{2} \left| \left(2\pi |\Psi(0)|^2 - 1 \right) \right|. \quad (C.13)$$

This last formula in case of wave functions (4) degenerates into trivial equality $0 = 0$

D: On the fluctuations of thermodynamic observables

Thermodynamic systems are macroscopic bodies composed by huge numbers of microscopic constituents (molecules and atoms). As whole bodies or through by their macroscopic parts such systems are described by so-called thermodynamic observables. The alluded observables are viewed as deterministic variables (in usual thermodynamics) respectively as stochastic quantities (in statistical physics). In the last view they are characterized by fluctuations (deviations from their deterministic values studied within usual thermodynamics). The mentioned fluctuations are investigated within the next conceptual frameworks: (a) phenomenological approach, (b) classical statistical mechanics, respectively (c) quantum statistical physics.

In phenomenological approach [68–72], proposed for the first time by Einstein, the respective fluctuations can be depicted briefly as follows. Let be a system of the mentioned kind, whose properties are described by a set of thermodynamic observables \mathbb{A}_j ($j = 1, 2, 3, \dots, n$). Each such observable \mathbb{A}_j is characterized by a global fixed value $\bar{\mathbb{A}}_j$, evaluable through the methods of deterministic usual thermodynamics. Then the fluctuations of observables \mathbb{A}_j should be discussed in terms of random variables $\mathcal{A}_j = \mathbb{A}_j - \bar{\mathbb{A}}_j$ ($j = 1, 2, \dots, n$), endowed with continuous spectra of values such as $\mathcal{A}_j \in (-\infty, +\infty)$. The random characteristics of variables \mathcal{A}_j , i.e. the fluctuations of observables \mathbb{A}_j , are depicted in phenomenological approach through the probability density $W = W(\vec{\mathcal{A}})$, where the vector $\vec{\mathcal{A}}$ signifies the set of all variables \mathcal{A}_j . Commonly for $W = W(\vec{\mathcal{A}})$ one uses distributions of Gaussian type. The mean value (expected) value $\langle \mathbb{A}_j \rangle_W$

and the random deviation $\delta_W \mathbb{A}_j$ of the observable \mathbb{A}_j are

$$\begin{aligned} \langle \mathbb{A}_j \rangle_W &= \int_{-\infty}^{+\infty} \mathbb{A}_j \cdot W(\vec{\mathcal{A}}) \cdot d\vec{\mathcal{A}}, \\ \delta_W \mathbb{A}_j &= \mathbb{A}_j - \langle \mathbb{A}_j \rangle_W = \mathcal{A}_j. \end{aligned} \quad (D.1)$$

Usually, the fluctuations of observables \mathbb{A}_j ($j = 1, 2, 3, \dots, n$) are characterized by a small number of numerical parameters evaluable through the random deviations $\delta_W \mathbb{A}_j$. Examples of such parameters are: dispersions $\langle (\delta_W \mathbb{A}_j)^2 \rangle_W = \langle (\mathcal{A}_j)^2 \rangle_W$ and their equivalents the standard deviations $\Delta_W \mathbb{A}_j = \sqrt{\langle (\delta_W \mathbb{A}_j)^2 \rangle_W}$, second order moments (correlations) $\langle \delta_W \mathbb{A}_j \cdot \delta_W \mathbb{A}_k \rangle_W$ ($j \neq k$) or even [72] higher order moments (correlations) $\langle (\delta_W \mathbb{A}_j)^r \cdot (\delta_W \mathbb{A}_k)^s \rangle_W$, ($r + s \geq 3$).

The correlations $\langle \delta_W \mathbb{A}_j \cdot \delta_W \mathbb{A}_k \rangle_W$ ($j, k = 1, 2, \dots, n$) constitute the components of a positive semi definite matrix. The respective components satisfy [70, 71] the following correlation formulas

$$\det \left[\langle \delta_W \mathbb{A}_j \cdot \delta_W \mathbb{A}_k \rangle_W \right] \geq 0, \quad (D.2)$$

where $\det [\alpha_{jk}]$ denote the determinant whose components are the quantities α_{jk} . Particularly for two thermodynamic observables $\mathbb{A}_1 = \mathbb{A}$ and $\mathbb{A}_2 = \mathbb{B}$ from (D.2) one obtains

$$\Delta_W \mathbb{A} \cdot \Delta_W \mathbb{B} \geq |\langle \delta_W \mathbb{A} \cdot \delta_W \mathbb{B} \rangle_W| \quad (D.3)$$

where $\Delta_W \mathbb{A} = \sqrt{\langle (\delta_W \mathbb{A})^2 \rangle_W}$ denotes the standard deviation of observable \mathbb{A} . Mathematically (in sense of probability theory) this last classical formula is completely analogous with the quantum UR (2).

Regarded in their detailed expressions the standard deviations like is $\Delta_W \mathbb{A}$ (introduced above) have an interesting generic property. Namely they appear as being in a direct and factorized dependence of Boltzmann's constant k_B . The respective dependence has the following physical significance. It is known the fact that, mathematically, for a given quantity the standard deviation indicates its randomness. This in the sense that the respective quantity is a random or, alternatively, a deterministic (non-random) variable according as the alluded deviation has a positive or null value. Therefore $\Delta_W \mathbb{A}$ can be regarded as an indicator of randomness for the thermodynamic observable \mathbb{A} . But, for diverse cases (of observables, systems and states), the deviation $\Delta_W \mathbb{A}$ has various expressions in which, apparently, no common element seems to be implied. Nevertheless such an element can be found out [20, 73] as being materialized by the Boltzmann's constant k_B . So, in Gaussian approximation within the framework of phenomenological theory of fluctuations one finds [20, 73]

$$(\Delta_W \mathbb{A})^2 = k_B \cdot \sum_{\alpha} \sum_{\beta} \frac{\partial \bar{\mathbb{A}}}{\partial \bar{\mathcal{X}}_{\alpha}} \cdot \frac{\partial \bar{\mathbb{A}}}{\partial \bar{\mathcal{X}}_{\beta}} \cdot \left(\frac{\partial^2 \bar{\mathbb{S}}}{\partial \bar{\mathcal{X}}_{\alpha} \partial \bar{\mathcal{X}}_{\beta}} \right)^{-1}. \quad (D.4)$$

Now note that, a kind of non-quantum formulas completely similar with (D.2) and (D.3), can be reported also

for the fluctuations of thermodynamic observables described in terms of classical statistical mechanics. In the respective terms the above phenomenological notations and relations can be transcribed formally as follows. Instead of random variables \mathcal{A}_j should to operate with the phase space ensemble denoted as μ of all coordinates and momenta of molecules/atoms which compose the thermodynamic system. Also instead of observables $\mathbb{A}_j = \overline{\mathbb{A}_j} + \mathcal{A}_j$ needs to be use the random functions of the form $\hat{\mathbb{A}}_j = \hat{\mathbb{A}}_j(\mu)$. Therewith the probability density $W = W(\vec{\mathcal{A}})$ should to be replaced with the statistical distribution function $w = w(\mu)$. Then, in terms of afore-said description of considered fluctuations, as example, can be written the relation

$$\Delta_w \mathbb{A} \cdot \Delta_w \mathbb{B} \geq |\langle \delta_w \mathbb{A} \cdot \delta_w \mathbb{B} \rangle_w| \quad (D.5)$$

which is completely similar with (D.3). Add here the observation that the standard deviations $\Delta_w \mathbb{A}$ and $\Delta_w \mathbb{B}$ from (D.5) have a factorization dependence on k_B of type (D.4), similarly with the case of quantities $\Delta_W \mathbb{A}$ and $\Delta_W \mathbb{B}$ from (D.3).

For describing the fluctuations of thermodynamic observables \mathbb{A}_j in framework of quantum statistical physics as probabilities carrier instead of phenomenological density $W = W(\vec{\mathcal{A}})$ should to use [73–76] the quantum density operator $\hat{\rho}$:

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle \langle \psi_k|. \quad (D.6)$$

Here $|\psi_k\rangle$ ($k = 1, 2, \dots$) denote the wave functions of pure states of system and p_k are the corresponding probabilities of the respective states. In the same framework the above mentioned random variables \mathcal{A}_j are substituted with the thermo-quantum operators $\hat{\mathbb{A}}_j$ ($j = 1, 2, \dots, n$). In framework of quantum statistical physics the mean value $\langle \mathbb{A}_j \rangle_\rho$ and random deviation $\delta_\rho \hat{\mathbb{A}}_j$ of observable \mathbb{A}_j are

$$\begin{aligned} \langle \mathbb{A}_j \rangle_\rho &= \sum_k p_k \langle \psi_k | \hat{\mathbb{A}}_j | \psi_k \rangle \\ &= \text{tr} \left(\sum_k p_k |\psi_k\rangle \langle \psi_k| \hat{\mathbb{A}}_j \right) = \text{tr} (\hat{\rho} \cdot \hat{\mathbb{A}}_j), \quad (D.7) \\ \delta_\rho \hat{\mathbb{A}}_j &= \hat{\mathbb{A}}_j - \langle \mathbb{A}_j \rangle_\rho. \end{aligned}$$

The deviations $\delta_\rho \hat{\mathbb{A}}_j$ can be used in description of numerical parameters of fluctuations for observables \mathbb{A}_j in the mentioned framework. As such parameters can be quoted: dispersions $\langle (\delta_\rho \hat{\mathbb{A}}_j)^2 \rangle_\rho$ and their equivalents standard deviations $\Delta_\rho \mathbb{A}_j = \sqrt{\langle (\delta_\rho \hat{\mathbb{A}}_j)^2 \rangle_\rho}$, second order moments (correlations) $\langle \delta_\rho \hat{\mathbb{A}}_j \cdot \delta_\rho \hat{\mathbb{A}}_k \rangle_\rho$ (where $j \neq k$) or even higher order moments $\langle (\delta_\rho \hat{\mathbb{A}}_j)^r \cdot (\delta_\rho \hat{\mathbb{A}}_k)^s \rangle_\rho$ (where $r + s \geq 3$).

In case of two thermodynamic observables \mathbb{A} and \mathbb{B} , regarded in framework of quantum statistical physics, can be introduced also a correlation relation similar with (D.3) and (D.5). Such a relation can be introduced as follows. For the

corresponding thermo-quantum operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ it is evidently true the relation

$$\sum_k p_k \left\langle \left(\delta_\rho \hat{\mathbb{A}} + i\lambda \delta_\rho \hat{\mathbb{B}} \right) \psi_k \mid \left(\delta_\rho \hat{\mathbb{A}} + i\lambda \delta_\rho \hat{\mathbb{B}} \right) \psi_k \right\rangle \geq 0 \quad (D.8)$$

where λ is an arbitrary real parameter. If in respect with the functions ψ_k the operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ satisfy the conditions of type (A.3) one can write

$$\begin{aligned} &\sum_k p_k \left\langle \left(\delta_\rho \hat{\mathbb{A}} + i\lambda \delta_\rho \hat{\mathbb{B}} \right) \psi_k \mid \left(\delta_\rho \hat{\mathbb{A}} + i\lambda \delta_\rho \hat{\mathbb{B}} \right) \psi_k \right\rangle \\ &= \sum_k p_k \left\langle \psi_k \mid \left(\delta_\rho \hat{\mathbb{A}} \right)^2 \mid \psi_k \right\rangle \\ &\quad + \lambda^2 \sum_k p_k \left\langle \psi_k \mid \left(\delta_\rho \hat{\mathbb{B}} \right)^2 \mid \psi_k \right\rangle \\ &\quad + i\lambda \sum_k p_k \left\langle \psi_k \mid \left(\delta_\rho \hat{\mathbb{A}} \cdot \delta_\rho \hat{\mathbb{B}} - \delta_\rho \hat{\mathbb{B}} \cdot \delta_\rho \hat{\mathbb{A}} \right) \mid \psi_k \right\rangle. \end{aligned} \quad (D.9)$$

Then from (D.8) it results the relation

$$\left\langle \left(\delta_\rho \hat{\mathbb{A}} \right)^2 \right\rangle_\rho + \lambda^2 \left\langle \left(\delta_\rho \hat{\mathbb{B}} \right)^2 \right\rangle_\rho + \lambda \langle i [\hat{\mathbb{A}}, \hat{\mathbb{B}}] \rangle_\rho \geq 0 \quad (D.10)$$

where $[\hat{\mathbb{A}}, \hat{\mathbb{B}}]$ denotes the commutator of operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$.

Because λ is an arbitrary real parameter from (D.10) one obtains the relation

$$\left\langle \left(\delta_\rho \hat{\mathbb{A}} \right)^2 \right\rangle_\rho \cdot \left\langle \left(\delta_\rho \hat{\mathbb{B}} \right)^2 \right\rangle_\rho \geq \frac{1}{4} \langle i [\hat{\mathbb{A}}, \hat{\mathbb{B}}] \rangle_\rho^2 \quad (D.11)$$

or the equivalent formula

$$\Delta_\rho \mathbb{A} \cdot \Delta_\rho \mathbb{B} \geq \frac{1}{2} \left| \langle i [\hat{\mathbb{A}}, \hat{\mathbb{B}}] \rangle_\rho \right|. \quad (D.12)$$

Now let us remind the fact that in quantum statistics the above discussed thermo-quantum quantities $\langle (\delta_\rho \hat{\mathbb{A}}_j)^2 \rangle_\rho$ and $\Delta_\rho \mathbb{A}$ are proved to be connected directly with a quantity from deterministic (simple thermodynamic) description of thermodynamic observables. The respective connection is due by the known fluctuation-dissipation theorem [76] which is expressed by the relation

$$\begin{aligned} \left\langle \left(\delta_\rho \hat{\mathbb{A}} \right)^2 \right\rangle_\rho &= (\Delta_\rho \mathbb{A})^2 \\ &= \frac{\hbar}{2\pi} \int_{-\infty}^{+\infty} \coth \left(\frac{\hbar\omega}{2k_B T} \right) \cdot \mathcal{X}''(\omega) \cdot d\omega. \end{aligned} \quad (D.13)$$

Here k_B = the Boltzmann's constant, \hbar = Planck's constant and T = temperature of the considered system. Also in (D.13) the quantity $\mathcal{X}''(\omega)$ denote the imaginary part of the susceptibility associated with the observable \mathbb{A} . Note that $\mathcal{X}''(\omega)$ is a deterministic quantity which is defined primarily in non-stochastic framework of macroscopic physics [77]. Due to the respective definition it is completely independent of both k_B and \hbar .

In the end of this Appendix the following conclusion may be recorded. All the relations (D.2), (D.3), (D.4), (D.10) and (D.11) are formulas regarding macroscopic fluctuations but not pieces which should be adapted to the UR–QMS philosophy requirements.

E: On the measurements of macroscopic fluctuations

The fluctuations parameters, defined above Appendix D, refer to the characteristics of intrinsic nature for the considered macroscopic systems. But in practical actions, for the same systems, one operates with global parameters, of double source (origin). A first source is given by the intrinsic properties of systems. A second source is provided by the actions of measuring devices. In such a vision a measurement can be regarded as an transmission process of information (referring to stochastic data). Consequently the data about the intrinsic properties of measured system appear as *input (in)* information while the global results of the corresponding measurement represent the *output (out)* information.

Here below we will appeal to the aforesaid vision for giving (as in [91, 107]) a theoretical model regarding the measurement of thermal fluctuations. The respective fluctuations will be considered in a phenomenological approach (see Appendix D). For simplicity let us consider a system characterized by a single macroscopic observable $\mathbb{A} = \bar{\mathbb{A}} - \mathcal{A}$, whose thermal fluctuations are impacted within the random variable \mathcal{A} having the spectrum $\mathcal{A} \in (-\infty, +\infty)$. The intrinsic fluctuations of \mathcal{A} is supposed to be described by the probability distribution $W_{in} = W_{in}(\mathcal{A})$ regarded as carrier of input-information. The results of measurements are depicted by the distribution $W_{out} = W_{out}(\mathcal{A})$ regarded as bearer of out-information. Then the measuring process may be symbolized as a transformation of the form $W_{in}(\mathcal{A}) \rightarrow W_{out}(\mathcal{A})$. If the measuring device is supposed to have stationary and linear characteristics, the mentioned transformation can be described as follows:

$$W_{out}(\mathcal{A}) = \int_{-\infty}^{+\infty} K(\mathcal{A}, \mathcal{A}') \cdot W_{in}(\mathcal{A}') \cdot d\mathcal{A}' \quad (E.1)$$

where $K(\mathcal{A}, \mathcal{A}')$ appears as a doubly stochastic kernel (in sense defined in [108]). This means that $K(\mathcal{A}, \mathcal{A}')$ satisfy the relations $\int_{-\infty}^{+\infty} K(\mathcal{A}, \mathcal{A}') d\mathcal{A} = \int_{-\infty}^{+\infty} K(\mathcal{A}, \mathcal{A}') d\mathcal{A}' = 1$.

Add here the fact that, from a physical perspective, the kernel $K(\mathcal{A}, \mathcal{A}')$ incorporates the theoretical description of all the characteristics of the measuring device. Particularly, for an ideal device which ensure $W_{out}(\mathcal{A}) = W_{in}(\mathcal{A})$, it must to have the expression $K(\mathcal{A}, \mathcal{A}') = \delta(\mathcal{A} - \mathcal{A}')$, where $\delta(\mathcal{X})$ denote the Dirac's δ -function of argument \mathcal{X} .

By means of distributions $W_\eta = W_\eta(\mathcal{A})$ ($\eta = in; out$) can be introduced the corresponding η -numerical-characteristics of thermal fluctuations of observable $\mathbb{A} = \bar{\mathbb{A}} + \mathcal{A}$. Such are the η -mean (expected) value $\langle \mathbb{A} \rangle_\eta$ and η - standard deviation $\Delta_\eta \mathbb{A}$ defined through the relations

$$\begin{aligned} \langle \mathcal{A} \rangle_\eta &= \int_{-\infty}^{+\infty} \mathcal{A} \cdot W_\eta(\mathcal{A}) \cdot d\mathcal{A}, \\ (\Delta_\eta \mathcal{A})^2 &= \left\langle (\mathcal{A} - \langle \mathcal{A} \rangle_\eta)^2 \right\rangle_\eta. \end{aligned} \quad (E.2)$$

The above considerations allow to note some observations about the measuring uncertainties (errors) regarding the fluctuating macroscopic observable \mathbb{A} . Firstly the $\eta = in$ -versions of the parameters (E.2) describe only the "intrinsic" properties of the measured system. Secondly the $\eta = out$ -variants of the same parameters incorporate composite information about the respective system and considered measuring device. That is why one can say that, in terms of the above discussions, the measuring uncertainties of observable \mathbb{A} should be described by the following error indicators (characteristics)

$$\begin{aligned} \mathcal{E} \{ \langle \mathbb{A} \rangle \} &= \langle \mathbb{A} \rangle_{out} - \langle \mathbb{A} \rangle_{in}, \\ \mathcal{E} \{ \Delta \mathbb{A} \} &= \Delta_{out} \mathbb{A} - \Delta_{in} \mathbb{A}. \end{aligned} \quad (E.3)$$

Observe here that because \mathbb{A} has stochastic characteristics for a relevant description of its measuring uncertainties it is completely insufficient the single indicator $\mathcal{E} \{ \langle \mathbb{A} \rangle \}$. An adequate minimal such description requires at least the couple $\mathcal{E} \{ \langle \mathbb{A} \rangle \}$ and $\mathcal{E} \{ \Delta \mathbb{A} \}$. For further approximations of errors caused by measurements can be taken into account [111] the higher order moments like the next ones

$$\mathcal{E} \{ \langle (\delta \mathbb{A})^n \rangle \} = \langle (\delta_{out} \mathbb{A})^n \rangle_{out} - \langle (\delta_{in} \mathbb{A})^n \rangle_{in} \quad (E.4)$$

where $\delta_\eta \mathbb{A} = \mathbb{A} - \langle \mathbb{A} \rangle$, $\eta = in, out$ and $n \geq 3$.

F: An exemplification for subsection 5.2

For presenting the announced exemplification we will refer to QMS of the energy for a particle of mass m , located in an infinite square well potential of width L [29]. The intrinsic state of the microparticle will be considered as being described by the *in*-wave function $\Psi_{in}(x) = \sum_{j=1}^n \alpha_j \cdot \varphi_j(x)$. Here $\varphi_j(x)$ denote the eigenfunctions associated to the energetic eigenvalues $a_j = E_j = \mathfrak{J} \cdot j^2$ where $\mathfrak{J} = (\hbar^2 \pi^2 / 2mL^2)$ and $j = 1, 2, 3, \dots$. In the considered *in*-wave function the quantities α_j are probability amplitudes corresponding to the eigenvalues E_j .

We will restrict our exemplification by taking into account only the following circumstances. So we take $n = 3$ as the upper value of the inner energy of the particle while for the amplitudes α_j we will consider the values which give $(|\alpha_j|^2) = (0.5 \ 0.4 \ 0.1)$.

Then the intrinsic characteristics of the particle energy are described by the next mean value and the standard deviation

$$\langle E \rangle_{in} = 3 \cdot \mathfrak{J}, \quad \Delta_{in} E = 2.45 \cdot \mathfrak{J}. \quad (F.1)$$

Accordingly with discussions from Subsection 5.2, for a particle in the mentioned intrinsic state, the measurement of energy can be described as follows. We need to define a model-expression for the matrix (M_{kj}) from (29). As a first example, we will consider a measurement done with a device endowed with flawed (FL) characteristics. Such devices, for

instance, can be associated with a matrix (M_{kj}) having the form

$$(M_{kj})_{FL} = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}. \quad (F.2)$$

Thus the outcomes of measurement will be characterized by probabilities $(|\beta_k|^2)_{FL} = (0.34 \ 0.38 \ 0.23)$. With such probabilities, the measurement outcomes for energy will be characterized by the next FL-expected-value and FL-standard-deviation

$$\langle E \rangle_{FL} = 3.98 \cdot \mathfrak{J}, \quad \Delta_{FL} E = 3.04 \cdot \mathfrak{J}. \quad (F.3)$$

Consequently, for the measurement described by (F.2), the error indicators (29) and (31) acquire the following FL-values

$$\mathcal{E}_{FL} \{ \langle E \rangle \} = 0.9 \cdot \mathfrak{J}, \quad \mathcal{E}_{FL} \{ \Delta E \} = 0.59 \cdot \mathfrak{J}. \quad (F.4)$$

If, for the above mentioned energy/particle, we want to describe a measurement done with a device having larger characteristics of accuracy (ACC) one can proceed as follows. In the spirit of the relations (31), for the matrix (M_{kj}) instead of the formula (F.2) we appeal, for example, to the following expression

$$(M_{kj})_{ACC} = \begin{pmatrix} 0.95 & 0.03 & 0.02 \\ -0.03 & 1.04 & -0.01 \\ 0.08 & -0.07 & 0.99 \end{pmatrix}. \quad (F.5)$$

So, for the probabilities associated to the outcomes of ACC-measurement, one obtains $(|\beta_k|^2)_{ACC} = (0.489 \ 0.4 \ 0.11)$. Associated to the respective probabilities, the considered ACC-measurement of energy is characterized by the next ACC-expected value and ACC-standard-deviation

$$\langle E \rangle_{ACC} = 3.088 \cdot \mathfrak{J}, \quad \Delta_{ACC} E = 2.52 \cdot \mathfrak{J}. \quad (F.6)$$

By comparing values from (F.6) with those from (F.1) one sees that the referred ACC-measurement is characterized by the following error indicators

$$\mathcal{E}_{ACC} \{ \langle E \rangle \} = 0.08 \cdot \mathfrak{J}, \quad \mathcal{E}_{ACC} \{ \Delta E \} = 0.07 \cdot \mathfrak{J}. \quad (F.7)$$

Finally, by comparing the results reported in relations (F.4) and (F.7), we can note the following remark. Within the above theoretical description of measurement, the error indicators (for both expected value and standard deviation) are much smaller in the case dealing with higher accuracy characteristics comparatively with the one regarding flawed features.

G: Illustrations for subsection 5.4

In order to illustrate the model discussed in Subsection 5.4, in connection with the description of QMS, let us present here an exercise taken by abbreviation from our article [20] (more computational details can be found in the respective article).

We will refer to a micro-particle of mass m having an one-dimensional motion along the x -axis. Its *in*-wave-function Ψ_{in} is taken of the form $\Psi_{in}(x) = |\Psi_{in}(x)| \cdot \exp \{ i\Phi_{in}(x) \}$ where

$$|\Psi_{in}(x)| \propto \exp \left\{ -\frac{(x-x_0)^2}{4\sigma^2} \right\}, \quad \Phi_{in}(x) = kx. \quad (G.1)$$

Here as well as below in other relations from this Appendix the explicit notations of normalization constants are omitted (they can be added easy by the interested readers). According to the wave function (G.1) the intrinsic features of the considered microparticle are described by the parameters x_0 , σ and k .

Through expressions (G.1), by means of formulas (38), it is simple to find the analytical expressions for probability density ρ_{in} and current j_{in} . As doubly stochastic kernels suggested in (40) we propose here the next two formulas

$$\Gamma(x, x') \propto \exp \left\{ -\frac{(x-x')^2}{2\gamma^2} \right\}, \quad (G.2)$$

$$\Lambda(x, x') \propto \exp \left\{ -\frac{(x-x')^2}{2\lambda^2} \right\}. \quad (G.3)$$

Here parameters γ and λ depict the characteristics of measuring devices/procedures. The values of the respective parameters are associated with an ideal measurement (when both γ and λ tend toward zero), respectively with a nonideal measurement (in cases when at least one of the two parameters is not-null).

Then, by using the procedures presented within Subsection 5.4, it is easy to find the *out*-entities ρ_{out} , j_{out} and Ψ_{out} . By using the respective entities together with the functions from (G.1) one can evaluate the *out* and *in* versions of mean (expected) values and standard deviations for observables of interest. The respective evaluations ensure estimations of the corresponding error indicators. So, for $\hat{x} = x =$ coordinate and $\hat{p} = -i\hbar\nabla_x =$ momentum as operators (observables) of interest, one obtains [20] the following error indicators

$$\mathcal{E} \{ \langle x \rangle \} = 0, \quad \mathcal{E} \{ \Delta x \} = \sqrt{\sigma^2 + \gamma^2} - \sigma, \quad (G.4)$$

$$\mathcal{E} \{ \langle p \rangle \} = 0,$$

$$\mathcal{E} \{ \Delta p \} = \hbar \left| \left[\frac{k^2(\sigma^2 + \gamma^2)}{\sqrt{(\sigma^2 + \lambda^2)(\sigma^2 + 2\gamma^2 - \lambda^2)}} - k^2 + \frac{1}{4(\sigma^2 + \gamma^2)} \right]^{\frac{1}{2}} - k \right|. \quad (G.5)$$

Let us now restrict in the wave function (G.1) to the situation when $x_0 = 0$ $k = 0$ and $\sigma = \sqrt{\frac{\hbar}{2m\omega}}$. Then (G.1) describe the ground state of a harmonic oscillator with $m =$ mass and $\omega =$ angular frequency. As observable of interest of such an oscillator we consider the energy described by the Hamiltonian

$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2}{2} x^2$. For the respective observable one finds

$$\langle H \rangle_{in} = \frac{\hbar\omega}{2}, \quad \Delta_{in}H = 0, \quad (G.6)$$

$$\langle H \rangle_{out} = \frac{\omega \left[\hbar^2 + (\hbar + 2m\omega\gamma^2)^2 \right]}{4(\hbar + 2m\omega\gamma^2)}, \quad (G.7)$$

$$\Delta_{out}H = \frac{\sqrt{2}m\omega^2\gamma^2(\hbar + m\omega\gamma^2)}{(\hbar + 2m\omega\gamma^2)}, \quad (G.8)$$

$$\mathcal{E}\{\langle H \rangle\} = \frac{m^2\omega^3\gamma^4}{\hbar + 2m\omega\gamma^2}, \quad (G.9)$$

$$\mathcal{E}\{\Delta H\} = \Delta_{out}H = \frac{\sqrt{2}m\omega^2\gamma^2(\hbar + m\omega\gamma^2)}{(\hbar + 2m\omega\gamma^2)}. \quad (G.10)$$

H: A more comprehensive description of measuring errors for random observables

In Subsections 5.2 and 5.4 or Appendices E, F and G, we have discussed the measuring errors for random observables of quantum respectively macroscopic nature. For description of that errors, were used as indicators only the lower order probabilistic parameters (moments and correlations). But those indicators give only first sequences, of limited value, for a global picture of the considered errors. A more comprehensive such a picture can be done in terms of informational entropies. Shortly, for the above discussed observables and errors, the suggested depiction can be illustrated as follows.

Firstly let us refer to the case of a macroscopic random observable \mathcal{A} whose measurements are outlined in Appendix E. The intrinsic characteristics (fluctuations) of \mathcal{A} are considered as being described by the probability distribution $W_{in} = W_{in}(\mathcal{A})$ regarded as carrier of input-information for measurements. The results of measurements are depicted by the distribution $W_{out} = W_{out}(\mathcal{A})$ associated with the out-information of measurements. The informational entropies \mathcal{H}_η ($\eta = in, out$) connected with the above noted distributions are defined through the formulas

$$\mathcal{H}_\eta(\mathcal{A}) = - \int_{-\infty}^{+\infty} W_\eta(\mathcal{A}) \cdot \ln [W_\eta(\mathcal{A})] \cdot d\mathcal{A}. \quad (H.1)$$

By taking into account the transformation (E.1), the main properties of the doubly stochastic kernel $K(\mathcal{A}, \mathcal{A}')$, as well

the formula $\ln(X) \leq X - 1$ one can write

$$\begin{aligned} & \mathcal{H}_{out}(\mathcal{A}) - \mathcal{H}_{in}(\mathcal{A}) \\ &= - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\mathcal{A} \cdot d\mathcal{A}' \cdot K(\mathcal{A}, \mathcal{A}') \cdot W_{in}(\mathcal{A}') \\ & \quad \cdot \ln \left[\frac{W_{out}(\mathcal{A})}{W_{in}(\mathcal{A}')} \right] \\ & \geq - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\mathcal{A} \cdot d\mathcal{A}' \cdot K(\mathcal{A}, \mathcal{A}') \cdot W_{in}(\mathcal{A}') \\ & \quad \cdot \left[\frac{W_{out}(\mathcal{A})}{W_{in}(\mathcal{A}')} - 1 \right] \\ &= 0. \end{aligned} \quad (H.2)$$

Therefore the errors specific of measurements for \mathcal{A} in its wholeness can be described through the comprehensive error indicator

$$\mathcal{E}\{\mathcal{H}(\mathcal{A})\} = \mathcal{H}_{out}(\mathcal{A}) - \mathcal{H}_{in}(\mathcal{A}) \geq 0. \quad (H.3)$$

This relationship shows that the measuring process can be described by a non-negative change in the informational entropy associated with the investigated observable. The situation when the respective change is null corresponds to the case of an ideal measurement (free of errors), mentioned otherwise in connection with the relationship (E.1).

Mostly, the macroscopic fluctuations described by the here used observable \mathcal{A} are investigated in the so-called Gaussian approximations. Then the entities $W_{in}(\mathcal{A})$ and $K(\mathcal{A}, \mathcal{A}')$ which appear in (E.1) are given by the following formulas

$$\begin{aligned} W_{in}(\mathcal{A}) &\propto \exp \left\{ -\frac{\mathcal{A}^2}{2a^2} \right\}, \\ K(\mathcal{A}, \mathcal{A}') &\propto \exp \left\{ -\frac{(\mathcal{A} - \mathcal{A}')^2}{2b^2} \right\}, \end{aligned} \quad (H.4)$$

where the explicit indication of normalization constants are omitted (the omission can be filled easily by interested readers). In the first formula from (H.4) a denotes the standard deviation of intrinsic fluctuations within the measured system. The symbol b in the second expression from (H.4) depicts the precision parameter of measuring device. Of course, for a scientifically acceptable measuring process, it must be considered that $b \ll a$.

In the alluded cases with Gaussian approximations the output distribution $W_{out}(\mathcal{A})$ has the form

$$W_{out}(\mathcal{A}) \propto \exp \left\{ -\frac{\mathcal{A}^2}{2(a^2 + b^2)} \right\}.$$

Then the comprehensive error indicator (H.3) becomes

$$\mathcal{E}\{\mathcal{H}(\mathcal{A})\} = \frac{1}{2} \ln \left(1 + \frac{b^2}{a^2} \right) \approx \frac{1}{2} \cdot \frac{b^2}{a^2}. \quad (H.5)$$

Now let us refer to the comprehensive informational depiction for measuring errors in cases of random quantum observables. We start the announced reference by discussing the case presented in Subsection 5.2, regarding the measurement of a quantum observable endowed with a discrete spectrum of eigenvalues. In the respective case the input and output data characterizing the measurement are depicted by the following corresponding probabilities

$$\mathcal{P}_{in}^j = |\alpha_j|^2, \quad \mathcal{P}_{out}^j = |\beta_j|^2, \quad (j = 1, 2, \dots, n). \quad (H.6)$$

These probabilities can be associated with the next information entropies

$$\mathcal{H}(\mathcal{P}_\eta) = - \sum_{j=1}^n \mathcal{P}_\eta^j \cdot \ln(\mathcal{P}_\eta^j), \quad (\eta = in, out). \quad (H.7)$$

Consequently, for an extensive description of measuring errors for the specified quantum observable, can be used the below comprehensive indicator

$$\mathcal{E}\{\mathcal{H}(\mathcal{P})\} = \mathcal{H}(\mathcal{P}_{out}) - \mathcal{H}(\mathcal{P}_{in}). \quad (H.8)$$

By taking into account the transformation (27), the basic properties of doubly stochastic matrix M_{jk} , plus the evident formula $\ln(X) \leq X - 1$, through some simple calculations (similar to those appealed in (H.2) and (H.3), one finds:

$$\mathcal{E}\{\mathcal{H}(\mathcal{P})\} \geq 0. \quad (H.9)$$

This formula corresponds to ideal or non-ideal measurements, in cases of equality respectively of inequality.

Note that, in cases of examples presented in Appendix F related with Subsection 5.2, the relation (H.8) takes the expressions

$$\begin{aligned} \mathcal{E}\{\mathcal{H}(\mathcal{P})_{FL}\} &= \mathcal{H}(|\beta_k|^2_{FL}) - \mathcal{H}(|\alpha_j|^2) \\ &= 0.131, \end{aligned} \quad (H.10)$$

$$\begin{aligned} \mathcal{E}\{\mathcal{H}(\mathcal{P})_{ACC}\} &= \mathcal{H}(|\beta_k|^2_{ACC}) - \mathcal{H}(|\alpha_j|^2) \\ &= 0.018. \end{aligned}$$

The above expressions correspond to measurements with characteristics of flawed respectively accurate types. The same expressions show that, even in informational-entropic approach, the measuring errors are higher in cases with flawed characteristics comparatively with the ones having accurate features.

Now let us note some things about the comprehensive description of measuring errors in cases approached in Subsection 5.4 and in Appendix G, regarding of quantum observables with continuous spectra. The corresponding measurements, depicted through the transformations (40), can be associated with the following informational entropies

$$\mathcal{H}_\eta(\rho) = - \int_{-\infty}^{+\infty} \rho_\eta(x) \cdot \ln(\rho_\eta(x)) \cdot dx, \quad (H.11)$$

$$\mathcal{H}_\eta(|j\rangle) = - \int_{-\infty}^{+\infty} |j_\eta(x)| \cdot \ln(|j_\eta(x)|) \cdot dx,$$

where $\eta = in, out$. Related with the above entropies can be introduced the next comprehensive error indicators

$$\mathcal{E}\{\mathcal{H}(\rho)\} = \mathcal{H}_{out}(\rho) - \mathcal{H}_{in}(\rho), \quad (H.12)$$

$$\mathcal{E}\{\mathcal{H}(|j\rangle)\} = \mathcal{H}_{out}(|j\rangle) - \mathcal{H}_{in}(|j\rangle).$$

Through some simple calculations (completely similar to the ones used in (H.2) and (H.3)) one finds that the error indicators (H.11) satisfy the relations

$$\mathcal{E}\{\mathcal{H}(\rho)\} \geq 0, \quad \mathcal{E}\{\mathcal{H}(|j\rangle)\} \geq 0 \quad (H.13)$$

These relations with equalities or inequalities refer to the cases of ideal respectively non-ideal measurements.

In particular case of measurement illustrated in Appendix G, associated with the doubly stochastic kernels (G.2) and (G.3), the error indicators (H.12) become

$$\mathcal{E}\{\mathcal{H}(\rho)\} = \ln \sqrt{\frac{\sigma^2 + \gamma^2}{\sigma^2}} \approx \frac{1}{2} \left(\frac{\gamma}{\sigma}\right)^2, \quad (H.14)$$

$$\mathcal{E}\{\mathcal{H}(|j\rangle)\} = \ln \sqrt{\frac{\sigma^2 + \lambda^2}{\sigma^2}} \approx \frac{1}{2} \left(\frac{\lambda}{\sigma}\right)^2.$$

The last expressions of these indicators imply the approximations $\gamma \ll \sigma$ and $\lambda \ll \sigma$, specific to the supposition that measuring devices have high accuracies. Of course that the cases with $\gamma = 0$ and $\lambda = 0$ depict the ideal measurements.

In the case of a harmonic oscillator, mentioned in the end of Appendix G, the first error indicator from (H.12) get the expression

$$\mathcal{E}\{\mathcal{H}(\rho)\} = \ln \sqrt{\frac{\hbar + 2m\omega\gamma^2}{\hbar}} \approx \frac{m\omega}{\hbar} \gamma^2. \quad (H.15)$$

Submitted on November 26, 2020

I: A private letter from the late scientist J. S. Bell to the present author**CERN**1985 Jan 29

Dear Dr Dumitru, thank you for your paper. I agree with what you say about the uncertainty principle: it has to do with the uncertainty in predictions rather than the accuracy of 'measurement'. I think in fact that the word 'measurement' has been so abused in quantum mechanics that it would be good to avoid it altogether.

I will send some papers, including (if I can find copies) those you request.

with best wishes

John Bell

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