The Exact Gödel Metric

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We demonstrate that Gödel's metric does not represent a model of universe as it is usually accepted in the standard literature. In fact, a close inspection shows that this metric as it stands is a very special case of a broader metric. Introducing a simple conformal transformation readily induces a pressure term on the right hand side of the Einstein's field equations which actually describe a peculiar perfect fluid. This term was wrongly interpreted by Gödel as the *ad hoc* cosmological constant required to sustain his model. Gödel's space-time can be thus regarded as a real physical system with no cosmological implication and it is relegated to the class of ordinary metrics. The emergence of the related closed time-like curves is not bound to a rotating universe as stated in all classical treatments and this fact naturally sheds new light on time travel feasibility considerations.

Notations

Space-time greek indices run from: α,β: 0, 1, 2, 3.
Space-time signature: -2.
$\varkappa$ is the Einstein constant.
We adopt here: $c = 1$.

1 The Gödel universe

1.1 General

In his original paper [1], Kurt Gödel has derived an exact solution to Einstein's field equations in which the matter takes the form of a shear/pressure free fluid (dust solution).

This universe is homogeneous but non-isotropic and it exhibits a specific rotational symmetry which allows for the existence of *close timelike curves* (*CTCs*). The Gödel space-time has a five dimensional group of isometries (G5) which is transitive. (An action of a group is transitive on a manifold (M,$\boldsymbol{g}$), if it can map any point of the manifold into any other points of M).

It admits a five dimensional *Lie algebra* of *Killing vector fields* generated by a time translation ∂_{x_0}, two spatial translations ∂_{x_1}, ∂_{x_2}, plus two further Killing vector fields:

$$\partial_{x_3} - x_2\partial_{x_2} \quad \text{and} \quad 2\mathrm{e}^{x_1}\partial_{x_0} + x_2\partial_{x_3} + \left(\mathrm{e}^{2x_1} - \frac{1}{2}\, {x_2}^2\partial_{x_2}\right).$$

The *Weyl tensor* of the *standard* Gödel solution has Petrov type D:

$$C^{\alpha\beta}{}_{\mu\nu} = R^{\alpha\beta}{}_{\mu\nu} + \frac{R}{3}\,\delta^{\alpha}{}_{[\mu}\delta_{\nu]}{}^{\beta} + 2\delta^{[\alpha}{}_{[\mu}R_{\nu]}{}^{\beta]}.$$

The presence of the non-vanishing Weyl tensor prevents the Gödel metric from being *Euclidean* whatever the coordinates transformations.

This is in contrast to the Friedmann-Lemaître-Robertson-Walker metric which can be shown to reduce to a conformal Euclidean metric, implying that its Weyl tensor is zero [2].

The Gödel universe is often dismissed because it implies a non zero cosmological term and also since its rotation would conflict with observational data.

In what follows, we are able to relax our demand that the Gödel metric be a description of an actual universe. This is achieved through a specific transformation which makes Gödel space-time an "ordinary" metric just as any other metrics currently derived in physics.

1.2 The basic theory

The classical Gödel line element is generically given by the interval

$$ds^2 = a^2\left[d{x_0}^2 - d{x_1}^2 + d{x_2}^2\frac{1}{2}\,\mathrm{e}^{2x_1} - d{x_3}^2 + 2\mathrm{e}^{x_1}(dx_0 dx_2)\right], \quad (1.1)$$

or equivalently:

$$ds^2 = a^2\left[-d{x_1}^2 - d{x_3}^2 - d{x_2}^2\frac{1}{2}\,\mathrm{e}^{2x_1} + (\mathrm{e}^{x_1}dx_2 + dx_0)^2\right]. \quad (1.2)$$

$a > 0$ is a constant.

The components of the metric tensor are:

$$(g_{\mu\nu})_{\mathrm{G}} = \begin{pmatrix} a^2 & 0 & a^2\mathrm{e}^{x_1} & 0 \\ 0 & -a^2 & 0 & 0 \\ a^2\mathrm{e}^{x_1} & 0 & a^2\frac{1}{2}\mathrm{e}^{2x_1} & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix},$$

$$(g^{\mu\nu})_{\mathrm{G}} = \begin{pmatrix} -a^2 & 0 & -a^{-2}2\mathrm{e}^{-x_1} & 0 \\ 0 & -a^2 & 0 & 0 \\ -a^{-2}2\mathrm{e}^{-x_1} & 0 & -a^{-2}2\mathrm{e}^{-2x_1} & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix}.$$

In this particular case, since only $\partial_1(g_{22})_{\mathrm{G}} \neq 0$ and $\partial_1(g_{02})_{\mathrm{G}} \neq 0$, the non-zero connection components are:

$$\Gamma^0{}_{01} = 1 \qquad \Gamma^0{}_{12} = \Gamma^1{}_{02} = \frac{1}{2}\,\mathrm{e}^{x_1}$$

$$\Gamma^1{}_{22} = \frac{1}{2}\,\mathrm{e}^{2x_1} \qquad \Gamma^2{}_{01} = -\mathrm{e}^{-x_1}$$

Those greatly simplify the Ricci tensor: $R_{\beta\gamma} = \partial_1\Gamma^1{}_{\beta\gamma} + \Gamma^1{}_{\beta\gamma} - \Gamma^\delta{}_{\alpha\beta}\Gamma^\alpha{}_{\delta\gamma}$ whose components reduce to:

$$R_{00} = 1, \qquad R_{22} = \mathrm{e}^{2x_1}$$

$$R_{02} = R_{20} = \mathrm{e}^{x_1}$$

The Gödel *unit vector* $\boldsymbol{u}$ of matter in the direction of the x_0 lines has the following components:

$$(u^\mu)_G = (a^{-1}, 0, 0, 0), \tag{1.3}$$

$$\left(u_\mu\right)_G = (a, 0, a e^{x_1}, 0), \tag{1.4}$$

hence:

$$R_{\mu\nu} = \left(u_\mu u_\nu\right)_G a^{-2}, \tag{1.5}$$

$$R = \left(u^\mu u_\mu\right)_G = a^{-2}. \tag{1.6}$$

In order to make his metric a compatible solution to Einstein's field equations, Gödel is led to introduce a cosmological constant Λ as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \varkappa \rho u_\mu u_\nu + g_{\mu\nu}\Lambda. \tag{1.7}$$

To achieve this compatibility, he then further sets:

$$a^{-2} = \varkappa\rho, \tag{1.8}$$

$$\Lambda = -\frac{1}{2} R = -\frac{1}{2a^2} = -\frac{1}{2}\varkappa\rho. \tag{1.9}$$

As primarily claimed by Gödel, its stationary space-time is homogeneous.

For every point A of the manifold $(\mathrm{M}, \boldsymbol{g}_G)$, there exists a one-parameter group of transformations of M carrying A into itself.

This means that $(\mathrm{M}, \boldsymbol{g}_G)$ has a rotational symmetry and matter rotates everywhere with a constant rotation velocity magnitude $\boldsymbol{\omega}_G$ orthogonal to $\boldsymbol{u}_G$.

Using the contravariant components:

$$(\omega^\alpha)_G = \left(0, 0, 0, \frac{\sqrt{2}}{a^2}\right), \tag{1.10}$$

one finds:

$$\omega_G = (g_{\alpha\mu}\omega^\alpha\omega^\mu)_G{}^{1/2} = \frac{a}{\sqrt{2}}. \tag{1.11}$$

With (1.8) this magnitude is:

$$\boldsymbol{\omega}_G = \left(\frac{1}{2}\varkappa\rho\right)^{1/2}. \tag{1.12}$$

A first glance at these constraints, readily reveals a fairly high degree of arbitrariness in the theory.

Finetuning the hypothetical constant Λ with the density of the universe (and the Ricci scalar) appears indeed as a somewhat dubious physical argument.

We shall see that those ill-defined assumptions are not required in order for the basic model to satisfy the field equations.

2 Gödel's model defined as a homogenous perfect fluid

2.1 Reformulation of Gödel's metric

We now make the assumption that a is slightly space-time variable and we set:

$$a^2 = e^{2U}. \tag{2.1}$$

The positive scalar $U(x_\mu) > 0$ will be explicited below.

The Gödel metric tensor components (1.2) are related to the fundamental metric tensor $\boldsymbol{g}$ by:

$$(g_{\mu\nu})_G = e^{2U} g_{\mu\nu}, \tag{2.2}$$

$$(g^{\mu\nu})_G = e^{-2U} g^{\mu\nu}, \tag{2.2 bis}$$

This means that the Gödel metric is now conformal:

$$ds^2 = e^{2U}\left[dx_0{}^2 - dx_1{}^2 + dx_2{}^2 \frac{1}{2} e^{2x_1} - dx_3{}^2 + 2e^{x_1}(dx_0 dx_2)\right]. \tag{2.3}$$

We are now going to see how the substitution (2.1) drastically changes the meaning of Gödel's limited theory.

2.2 Relativistic analysis of a neutral homogeneous perfect fluid

2.2.1 The geodesic differential system

Let us consider the manifold $(\mathrm{M}, \boldsymbol{g})$, on which is defined a vector tangent to the curve C in local coordinates:

$$\dot{x}^\alpha = \frac{dx^\alpha}{d\zeta}, \quad \text{where } \zeta \text{ is an affine parameter.}$$

In these coordinates we consider the scalar valued function $f(x^\alpha, \dot{x}^\alpha)$ which is homogeneous and of first degree with respect to $\dot{x}^\alpha$.

To the curve C joining the point x_1 to x_2, one can always associate the integral $\mathcal{A}$ such that

$$\mathcal{A} = \int_{\zeta_1}^{\zeta_2} f(x^\alpha, \dot{x}^\alpha) d\zeta = \int_{x_1}^{x_2} f(x^\alpha, \dot{x}^\alpha) dx^\alpha. \tag{2.4}$$

We now want to evaluate the variation of $\mathcal{A}$ with respect to the points ζ_1 and ζ_2:

$$\delta\mathcal{A} = f\,\delta_{\zeta_2} - f\,\delta_{\zeta_1} - \int_{\zeta_1}^{\zeta_2} \delta f\, d\zeta.$$

Classically we know that:

$$\int_{\zeta_1}^{\zeta_2} \delta f\, d\zeta = \left[\frac{\partial f}{\partial \dot{x}^\alpha}\delta x^\alpha\right] - \int_{\zeta_1}^{\zeta_2} E_\alpha \delta x^\alpha d\zeta,$$

where E_α is the first member of the *Euler equations* associated with the function f.

With E_α as the components of $\boldsymbol{E}$, we infer the expression

$$\delta\mathcal{A} = [w(\delta)]_{x_2} - [w(\delta)]_{x_1} - \int_{\zeta_1}^{\zeta_2} \boldsymbol{E}\delta x\, d\zeta, \tag{2.5}$$

where $[w(\delta)]$ has the form:

$$[w(\delta)] = \left(\frac{\partial f}{\partial \dot{x}^\alpha}\right)\delta x^\alpha - \left(x^\alpha \frac{\partial f}{\partial \dot{x}^\alpha} - f\right)\delta\zeta.$$

Due to the homogeneity of f, it reduces to:

$$w(\delta) = \left(\frac{\partial f}{\partial \dot{x}^\alpha}\right)\delta x^\alpha.$$

Let us apply the above results to the function

$$f = \mathrm{e}^U \frac{ds}{d\zeta} = \mathrm{e}^U\left(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right)^{1/2},$$

where e^U is defined everywhere on $(\mathrm{M},\boldsymbol{g})$.

Between two points x_1 and x_2, of $(\mathrm{M},\boldsymbol{g})$ connected by a time-like curve we have the correspondence:

$$s' = \int_{x_1}^{x_2} \mathrm{e}^U ds = \int_{x_1}^{x_2} \mathrm{e}^U\left(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right)^{1/2}. \tag{2.6}$$

We first differentiate $f^2 = \mathrm{e}^{2U}\left(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right)$ with respect to $\dot{x}^\alpha$ and x^α:

$$f\frac{\partial f}{\partial \dot{x}^\alpha} = \mathrm{e}^{2U} g_{\alpha\beta}\dot{x}^\beta, \tag{2.7}$$

$$f\frac{\partial f}{\partial x^\alpha} = \mathrm{e}^U\left(g_{\beta\mu}\dot{x}^\beta\dot{x}^\mu\right)^{1/2} \times\left[\partial_\alpha \mathrm{e}^U\left(g_{\beta\mu}\dot{x}^\beta\dot{x}^\mu\right)^{1/2} + \frac{1}{2}\,\mathrm{e}^U\partial_\alpha\left(g_{\beta\mu}\dot{x}^\beta\dot{x}^\mu\right)\right]. \tag{2.8}$$

We now choose s as the affine parameter ζ of the curve C, so the vector $\dot{x}^\beta$ is here regarded as the unit vector u^β tangent to C.

Equations (2.7) and (2.8) then reduce to the following:

$$\frac{df}{d\dot{x}^\beta} = \mathrm{e}^U u_\beta, \tag{2.9}$$

$$\frac{df}{dx^\beta} = \partial_\beta \mathrm{e}^U + \frac{1}{2}\,\mathrm{e}^U\partial_\beta(g_{\alpha\mu})u^\alpha u^\mu,$$

$$\frac{df}{dx^\beta} = \partial_\beta \mathrm{e}^U + \mathrm{e}^U\Gamma_{\alpha\beta,\mu}u^\alpha u^\mu. \tag{2.10}$$

The $\Gamma_{\alpha\beta,\mu}$ are here the Christoffel symbols of the first kind.

Expliciting the Euler equations $f(x^\alpha,du^\alpha)$:

$$E_\beta = \frac{d}{ds}\frac{\partial f}{\partial u^\beta} - \frac{\partial f}{\partial x^\beta}, \tag{2.11}$$

we get:

$$E_\beta = \frac{d}{ds}\left(\mathrm{e}^U u_\beta\right) - \mathrm{e}^U\left(\Gamma_{\alpha\beta,\mu}u^\alpha u^\mu\right) - \partial_\beta \mathrm{e}^U,$$

$$E_\beta = \mathrm{e}^U\left(u^\mu\partial_\mu u_\beta - \Gamma_{\alpha\beta,\mu}u^\alpha u^\mu\right) - \partial_\alpha \mathrm{e}^U\left(\delta_\beta{}^\alpha - u^\alpha u_\beta\right),$$

$$E_\beta = \mathrm{e}^U\left[\left(u^\mu\nabla_\mu u_\beta\right) - \partial_\beta U - \partial_\alpha U\left(\delta_\beta{}^\alpha - u^\alpha u_\beta\right)\right]. \tag{2.12}$$

Equation (2.5) becomes:

$$\delta\mathcal{A} = [w(\delta)]_{x_2} - [w(\delta)]_{x_1} - \int_{x_1}^{x_2}\langle \boldsymbol{E}\delta x\rangle ds, \tag{2.13}$$

where locally: $w(\delta) = \mathrm{e}^U u_\alpha dx^\alpha$.

When the curve C varies between two *fixed* points x_1 and x_2 the local variations $[w(\delta)]_{x_2}$ and $[w(\delta)]_{x_1}$ vanish. Therefore applying the variational principle to (2.13) simply leads to:

$$\delta\mathcal{A} = -\int_{x_1}^{x_2}\langle \boldsymbol{E}\delta x\rangle ds = 0, \tag{2.14}$$

from which we infer $\boldsymbol{E} = 0$, i.e., from (2.12):

$$u^\mu\nabla_\mu u_\beta - \partial_\alpha U\left(\delta_\beta{}^\alpha - u^\alpha u_\beta\right) = 0 \qquad (\text{since } \mathrm{e}^U \neq 0). \tag{2.15}$$

The equation (2.15) is formally identical to the differential system obeyed by the flow lines of a perfect fluid of density ρ with an equation of state $\rho = f(P)$ (*see Appendix*):

$$T_{\mu\beta} = (\rho + P)u_\mu u_\beta - P g_{\mu\beta}. \tag{2.16}$$

These flow lines are thus timelike geodesics of the conformal metric to $(\mathrm{M},\boldsymbol{g})$ according to (2.6):

$$s' = \int_{s_1}^{s_2}\mathrm{e}^U ds, \tag{2.17}$$

with

$$U = \int_{P_1}^{P_2}\frac{dP}{\rho + P}. \tag{2.18}$$

All along the curve segment (s'), the pressure is varying between two endpoints s_1 and s_2 which correspond to the values P_1 and P_2.

One can find similar conclusions in [3, 4].

The positive scalar e^U accounts for the *relativistic fluid index* [5].

2.2.2 The Gödel interpretation

The tensor (2.16) can be equivalently written:

$$T_{\mu\beta} = \rho u_\mu u_\beta - P h_{\mu\beta}, \tag{2.19}$$

with the *projection tensor*:

$$h_{\mu\beta} = g_{\mu\beta} - u_\mu u_\beta. \tag{2.20}$$

The cosmological term can then be re-introduced by setting

$$P = -\frac{\Lambda}{\varkappa}, \tag{2.21}$$

yielding the model which Gödel simply focused on.

Finally, by letting a be a *conformal factor*, we see that Gödel's metric (2.3) is simply the solution of the field equations with a variable pressure term as per:

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = \varkappa\left(\rho u_\mu u_\beta - P h_{\mu\beta}\right). \tag{2.21bis}$$

The cosmological "constant" Λ is thus no longer this arbitrary ingredient required to sustain the Gödel model and so are the constraints (1.8) and (1.9).

2.3 The Gödel rotation

2.3.1 Vorticity of the fluid

We just showed that Gödel space-time should be likened to a perfect fluid.

The time-like 4-vector u_α is everywhere tangent to the flow lines of this fluid.

The covariant derivative $u_{\alpha;\mu}$ may be expressed in a invariant manner in terms of tensor fields which describe the kinematics of the congruence of curves generated by u^α.

In Gödel's case, the *shear tensor* $\sigma_{\alpha\mu}$ vanishes:

$$\sigma_{\alpha\mu} = u_{(\alpha;\mu)} - \frac{1}{3}\,\theta h_{\alpha\mu} + \mathring{u}_{(\alpha}u_{\mu)} = 0, \tag{2.22}$$

where θ is the *expansion scalar*:

$$\theta = u^\alpha{}_{;\alpha}. \tag{2.22bis}$$

$\mathring{u}_\alpha$ is the *acceleration vector* of the flow lines:

$$\mathring{u}_\alpha = u_{\alpha;\mu}u^\mu. \tag{2.22ter}$$

For a perfect fluid, this acceleration is shown to be (*see Appendix*):

$$\mathring{u}_\alpha = \partial_\alpha U. \tag{2.23}$$

Besides $\mathring{u}_\alpha$ and θ, the shearless fluid is characterized by the *vorticity tensor*:

$$\omega_{\alpha\mu} = h_\alpha{}^\sigma h_\mu{}^\nu u_{[\sigma;\nu]} = u_{[\alpha;\mu]} + \mathring{u}_{[\alpha}u_{\mu]}, \tag{2.24}$$

from which is derived the vorticity *4-vector* $\boldsymbol{\omega}$ of the flow lines of the fluid.

The $\boldsymbol{\omega}$-components are known to be: [6]

$$\omega^\beta = \frac{1}{2}\,\eta^{\beta\gamma\sigma\rho}u_\gamma\omega_{\sigma\rho}, \tag{2.25}$$

with the Levi-Civita tensor: $\eta^{\beta\gamma\sigma\rho} = -g^{-1/2}\cdot\varepsilon^{\beta\gamma\sigma\rho}$.

The kinematic quantities $\omega_{\alpha\mu}$ and ω_α are completely orthogonal to u^μ , i.e.,

$$\omega_{\alpha\mu}u^\mu = 0, \quad \mathring{u}_\alpha u^\mu = \omega_\alpha u^\mu = 0.$$

(Shear free flows of a perfect fluid associated with the Weyl tensor have been extensively investigated by A. Barnes, Classical General Relativity. proc. Cambridge, 1984).

2.3.2 Conformal transformations

All above results can be easily extended to the conformal manifold $(\mathrm{M},\boldsymbol{g}')$ by applying the covariant derivative $(\nabla_\mu)'$ formed with the *conformal connection coefficients*:

$$\left(\Gamma^\gamma{}_{\alpha\beta}\right)' = \Gamma^\gamma{}_{\alpha\beta} + 2\delta^\gamma{}_{(\alpha}U_{,\beta)} - g_{\alpha\beta}U^{,\gamma}. \tag{2.26}$$

One also defines the unit 4-vector $\boldsymbol{w}$ of the fluid on the conformal metric $(ds^2)'$ as:

$$w^\mu = \mathrm{e}^U u^\mu, \tag{2.27}$$

$$w_\beta = \mathrm{e}^{-U}u_\beta. \tag{2.28}$$

In this case, the differential system of the flow lines w^μ admits the *relative integral invariant* in the sense of Poincaré [7]:

$$\int \Omega = \int w_\beta\,\delta x^\beta. \tag{2.29}$$

Denoting by $d\Omega$ the exterior differential of the form Ω, we have in local coordinates:

$$d\Omega = dw_\beta \wedge dx^\beta = \frac{1}{2}\Big[\partial_\beta w_\alpha - \partial_\alpha w_\beta\Big]dx^\beta \wedge dx^\alpha. \tag{2.30}$$

To the form $d\Omega$ is associated the antisymmetric tensor of components:

$$\omega_{\beta\alpha} = \partial_\beta w_\alpha - \partial_\alpha w_\beta. \tag{2.31}$$

It is easy to verify that these components coincide with the vorticity tensor components defined by (2.24). Unlike the vorticity tensor $\omega_{\beta\alpha}$, the vorticity vector ω^β does not remains the same upon the conformal transformations (2.27)–(2.28).

2.3.3 Application to the Gödel model

On the modified Gödel manifold $(\mathrm{M},\boldsymbol{g}_\mathrm{G})$, the components of the unit 4-vector $\boldsymbol{w}_\mathrm{G}$ tangent to world lines of matter (1.3) (1.4) are here:

$$(w^\mu)_\mathrm{G} = \mathrm{e}^U(u^\mu)_\mathrm{G} = \mathrm{e}^U(1,0,0,0), \tag{2.32}$$

$$\left(w_\beta\right)_\mathrm{G} = \mathrm{e}^U\left(u_\beta\right)_\mathrm{G} = \mathrm{e}^{-U}(1,0,\mathrm{e}^{x_1},0). \tag{2.33}$$

Notice that the contravariant components $(u^\mu)_\mathrm{G}$ are all constant.

In this *particular case*, according to (2.23), one has

$$(\mathring{u}_\alpha)_\mathrm{G} = \partial_\alpha U = 0, \quad \text{i.e., } U \text{ is constant.}$$

By concatenation, the conformal factor $\exp U$ reduces to a constant and coincides with Gödel's choice $a = const$.

So the vorticity magnitude of the fluid's matter remains as in the initial theory:

$$\omega_\mathrm{G} = \left(g_{\alpha\mu}\omega^\alpha\omega^\mu\right)_\mathrm{G}^{1/2} = \frac{a}{\sqrt{2}}. \tag{2.34}$$

On the other hand, we note that the covariant components of the velocity $(u_\beta)_\mathrm{G}$ are not all constant.

This means that the conformal geodesics principle holds within our theory.

In other words, we clearly see that Gödel's proposed solution is only a (very limited) special case (contravariant velocity components) which therefore reveals a patent lack of generality.

Therefore, Gödel's theory ought to be embedded in a broader scheme implying a conformal metric $\left(ds^2\right)'$ as we inferred above.

Note: one of the *Kretschmann scalar* is an invariant only for ω_G : $R_{\alpha\beta\gamma\delta}T^{\alpha\beta\gamma\delta} = 12\omega_G{}^4$.

2.4 Chronal horizon

With Gödel one defines new (cylindrical) coordinates (t,r,ϕ,y) by setting:

$$e^{x_1} = \cosh 2r + \cos\phi \sinh 2r, \tag{2.35}$$

$$x_2 e^{x_1} = \sqrt{2}\sin\phi \sinh 2r, \tag{2.36}$$

$$\tan\frac{1}{2}\left[\phi + \frac{x-2t}{\sqrt{2}}\right] = e^{-2r}\tan\frac{\phi}{2}, \tag{2.37}$$

$$2z = x_3. \tag{2.38}$$

Within the framework of our theory, these coordinates lead to the line element:

$$ds^2 = 4e^{2U}(x)\Big[dt^2 - dr^2 - dz^2 + \\ +\left(\sinh^4 r - \sinh^2 r\right)d\phi^2 + 2\sqrt{2}\sinh^2 r\, d\phi\, dt\Big]. \tag{2.39}$$

This metric still exhibits the rotational symmetry of the solution about the axis $r = 0$, since we clearly see that the components of the metric tensor do not depend on ϕ.

For $r \geqslant 0$, we have: $0 \leqslant \phi \leqslant 2\pi$. If a curve r_G is defined by: $\sinh^4 r = 1$, that is

$$r_G = \ln\left(1+\sqrt{2}\right), \tag{2.40}$$

then any curve $r > \ln(1+\sqrt{2})$, i.e. $(\sinh^4 r - \sinh^2 r) > 0$ materialized in the "plane" $t = const.$ (or zero t), is a *closed timelike curve*.

The radius r_G referred to as the *Gödel radius* induces a *light-like curve* or *closed null curve*, where the light cones are tangential to the plane of constant (or zero) t.

The photons trajectories reaching this radius are closing up, therefore r_G constitutes a *chronal horizon* beyond which an observer located at the origin ($r = 0$) cannot detect them.

With increasing $r > r_G$ the light cones continue to keel over and their opening angles widen until their future parts reach the negative values of t.

In this *achronal domain*, any trajectory is a closed time-like curve and s' is extended over a full cycle.

As a result, the integral U performed over the closed path has no endpoints and is thus expressed in the form:

$$U = \int\left[\frac{dP}{\rho + P}\right] + const. \tag{2.41}$$

However, the pressure P which is fluctuating along the closed path remains at the same averaged value for the whole cycle and may be then regarded as globally constant.

In this case, the first term in the r.h.s. of (2.35) vanishes implies $U = const.$, and the conformal factor $(\exp U)$ may coincide again with Gödel's choice $a = const.$

Therefore, for $r > r_G$, the acceleration of flow lines of matter is always zero whatever the components of w_G. Because of this, all closed timelike curves can no longer be derived from the geodesic principle calculation developed above.

By introducing the pressure in the Gödel model, we clearly put in evidence the difference between the geodesics and the closed time-like curves.

This was mathematically outlined in [8] but no explanation was provided as *why* this difference arises.

Conclusion

When Gödel wrote down his metric he was led to introduce a distinctive constant factor a in order to re-transcript the field equations with a cosmological constant along with additional constraints.

Our theory is free of all these constraints and moreover it provides a physical meaning to the a term. Inspection shows that by substituting a conformal factor to the constant a induces the field equations with a pressure like term which was wrongly interpreted by Gödel as the cosmological constant of the universe.

In fact, he empirically assembled the pieces of the constant matter density and curvature scalar in order to conveniently cope with the field equations precisely written with the cosmological constant.

In contrast, the reconstructed Gödel metric is here a straightforward solution to these equations and as such it can be reproduced like any other metric without referring to any cosmological model whatsoever.

The metric still exhibits a rotation which allows for the existence of *close timelike curves (CTCs)* since the light cone opens up and tips over, as the Gödel's circular coordinate radius increases within the cylindrical coordinates representation.

It seems that the first model exhibiting this property was pioneered by the German mathematician C. Lanczos in 1924 [9], and later rediscovered in a new form by the dutch physicist W. J. Van Stockum in 1937 [10].

However, the existence of *CTCs* satisfying the Einstein's equations remained so far a stumbling block for most of physicists because it should imply the possibility to travel back and forth in time.

The time travel possibility, was quoted as a pure mathematical "exercise" unrealistic in nature because it was deemed to describe a hypothetical universe contradicting the standard model in expansion as we observe it. Moreover, defining an absolute time is not readily applicable in Gödel space-time.

In here, the cards are now somewhat reshuffled: the Gödel model does not describe any sort of universe and the relevant metric can be applied as any other metrics like for example the Schwarzschild, the Kerr or the Alcubierre's ones.

Under these circumstances, why not considering the Gödel model as a potential time machine?

A typical example of such possible time machine is given by the cylinder system elaborated by the American physicist F. J. Tipler in 1974 [11].

It describes an infinitely long massive cylinder spinning along its longitudinal axis which gives rise to the "frame dragging" effect. If the rotation rate is fast enough the light cones of objects in cylinder's vicinity become tilted. Tipler suggested that a finite cylinder might also produce *CTCs* which was objected by Hawking who argued that any finite region containing *CTCs* would require negative energy density produced by a so-called "exotic matter" which violates all *energy conditions* [12].

The same kind of negative energy is needed to sustain a coupled system of *Lorentzian wormholes* designed to create a time machine as suggested in [13].

In all cases, feasibility and related causality paradoxes seemed to have been killed once for good by Hawking through a specific vacuum fluctuations mechanism that impedes any attempt to travel in the past [14].

Several authors have however recently challenged if not rejected this statement [15, 16].

These constraints do not apply in the present theory.

For a thorough study covering *CTCs* questions one can refer to [17, 18].

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Appendix

In a *holonomic frame* defined on (M,$\boldsymbol{g}$), the unit vectors are normalized so that:

$$g_{\mu\nu}u^{\mu}u^{\nu} = g^{\mu\nu}u_{\mu}u_{\nu} = 1. \tag{A.1}$$

By differentiating we get:

$$u^{\mu}\nabla_{\nu}u_{\mu} = 0. \tag{A.2}$$

Let us consider the following tensor which describes a homogeneous perfect fluid with density ρ and with pressure P:

$$T_{\mu\nu} = (\rho + P)u_{\mu}u_{\nu} - Pg_{\mu\nu}. \tag{A.3}$$

The conservation equations are written:

$$\nabla_{\mu}[(\rho + P)u^{\mu}u_{\nu}] = \nabla_{\mu}\left(P\delta_{\nu}{}^{\mu}\right). \tag{A.4}$$

Setting the vector K_ν such that

$$(\rho - P)K_{\nu} = \nabla_{\mu}\left(P\delta_{\nu}{}^{\mu}\right), \tag{A.5}$$

$$\nabla_{\mu}[(\rho - P)u^{\mu}u_{\nu}] = (\rho + P)K_{\nu}, \tag{A.6}$$

$$\nabla_{\mu}[(\rho + P)u^{\mu}]u_{\nu} + (\rho + P)u^{\mu}\nabla_{\mu}u_{\nu} = (\rho + P)K_{\nu}. \tag{A.7}$$

Multiplying through with u^ν, and taking into account (A.2), we obtain after dividing by $(\rho + P)$:

$$u^{\mu}\nabla_{\mu}u_{\nu} = \left(g_{\mu\nu} - u_{\mu}u_{\nu}\right)K^{\mu} = h_{\mu\nu}K^{\mu}. \tag{A.8}$$

The flowlines everywhere tangent to the vector u^μ are determined by the differential equations (A.8).

K^μ only depends on x^μ and since: $h_{\mu\nu}K^{\mu} = K_{\nu} = \partial_{\nu}\dfrac{P}{\rho - P}$, we set

$$K_{\nu} = \partial_{\nu}U, \tag{A.9}$$

with

$$U = \int \frac{dP}{\rho + P}. \tag{A.10}$$

When the fluid pressure is function of the density, the 4-vector $\partial_\nu U$ is regarded as the *4-acceleration vector* $\mathring{u}_\nu$ of the flow lines given by the pressure gradient orthogonal to those lines [19, p.70].