Are Tensorial Affinities Possible?

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This short paper is an extraction from our previous work [1], the purpose of which is to make clear that it is very much possible to use Weyl's idea [2] of a conformal metric to achieve tensorial affinities. We are of the strong view that this is very important as it is predominantly assumed that this is not possible. We want to dispel this myth once and for all.

Nature is pleased with simplicity.

Sir Isaac Newton (1642-1727)

1 Introduction

The purpose of this article is to present in a much simpler and succinct form, the ideas presented in our first installation [1] on an attempt to bring the gravitational force and the other forces of Nature (the Electromagnetic, Weak and the Strong Nuclear force) into unity with all the other forces and, as well, unity of all the forces with Quantum Mechanics. For clarity's sake, we have herein removed most of the intricate mathematical and philosophical details found in [1]. We hope this abridged version will make clear to our readers what it is we have done in paper [1].

Further, the purpose and motivation of the present paper has been propelled by one of our favourite Weylian blogger and American physicist – Dr. William O. Straub. He posted on his blog-site* on 28 February 2022 an interesting article entitled: *I'm Still Rooting for the Underdog*. In his article, Dr. Straub expresses his justified frustration on the lack of progress in the search for darkmatter and wonders if it is not time for physicists to abandon this idea/concept and seriously consider much more seriously already existing alternative theories to darkmatter – e.g. Milgrom's *Modified Newtonian Gravity* (MoND) [3–5]. Dr. Straub's frustration is not his alone, it is shared by a plethora of physicists.

To prepare his reader(s) for the conclusion that he seeks, in the introduction of his article, Dr. Straub talks of *perpetual motion machines* and the *luminiferous aether* – i.e. concepts that were once thought to have a direct relation with reality but were eventually found to be worthless/non-physical and were thus abandoned by mainstream science and these ideas are not expected to re-appear anytime soon in mainstream science.

Amongst the many alternative ideas to darkmatter, Dr. Straub considers the subtle flaws in Einstein [6]'s General Theory of Relativity (GTR) and wonders if Weyl's [2] supposed failed unified theory of gravitation and electromagnetism holds any hope as an alternative theory to darkmatter. In the penultimate of his article: of Weyl's [2] theory, Dr. Straub had this to say:

> To me, there is one glaring flaw in Einstein's theory, which is its noninvariance with respect to conformal transformations. Weyl also saw this as a flaw, and he showed us a possible way to fix it.

After reading Dr. Straub's article on the morning of 1 March 2022, I was particularly struck by the first sentence in his statement. I immediately wrote to him saying

> I must say, I hold the same view and like Einstein [7, 8], Schrödinger [9–11], *etc*, I believe this requires that the affinities be tensors. I have worked out a new theory that is just that – I am sure I have sent this to you before.

Rather swiftly, Dr. Straub responded to my email by saying: …*Turning the connections into true tensors will be a tough job, and I'm inclined to believe it can't be done.* His response challenged me to write a much simpler version of the idea that I used in [1], i.e. the idea of obtaining tensorial affinities. This is what we present below and I hope it is much clearer than it is presented in [1].

2 Riemann geometry

From a viewpoint of geometry, Einstein [6]'s greatest and most beautiful masterpiece, the GTR, has its rock solid foundations anchored in Riemann Geometry† (RG). Fundamental in RG are the affine connections (Christoffel three symbols), namely:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\delta\lambda} \left(g_{\delta\mu,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta} \right) . \tag{1}$$

Their topological defect, insofar as the GTR is concerned, is that these affine connections are not tensors, as they transform in the following manner:

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \Gamma^{\lambda}_{\mu\nu} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial x^{\mu'} \partial x^{\nu'}} . \tag{2}$$

*`http://www.weylmann.com/aftermath.shtml`, visited on 5 Mar. 2022 @ 16h18 GMT+2

†I shall assume that the reader(s) knows very well Riemann geometry together with its symbols as commonly presented in the textbooks. Hence, we will not explain these but assume the reader(s) is/are *in sync* with us.

The first term on the right hand side of (2) has the characteristic transformational properties of a tensor while the second term destroy the *to-be* tensorial character of the affine. If this second term on the right hand side of (2) was not present, the affine would surely be a tensor. These affine connections present a problem when it comes to the geodesic equation of motion, namely:

$$\frac{d^2x^\lambda}{ds^2} - \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,. \tag{3}$$

Because of the nature of the non-tensorial affine connection $\Gamma^\lambda_{\mu\nu}$, this geodesic (3) of motion does not holdfast – in the truest sense – to the depth of the letter and essence of the philosophy deeply espoused and embodied in Einstein's *Principle of Relativity* (PoE) [12], namely that physical laws must require no special set of coordinates where there are to be formulated.

The non-tensorial nature of the affine connections require that the equation of motion must first be formulated in special kind of coordinate systems known as a *geodesic coordinate systems**, yet the PoE forbids this. This problem has never been adequately addressed in the GTR. In order to appreciate that this indeed is a real problem, one can for example consider the fact that affinities in the GTR represent forces. A force has no relative sense of existence either by way of a coordinate transformation or a transformation between reference systems – yet, the affine connection speaks to the construction of this seemingly non-physical scenario.

That is to say: if a force exists (i.e. $\Gamma^\lambda_{\mu\nu} \neq 0$) in one coordinate system, it must exist in any arbitrary coordinate system (i.e. $\Gamma^{\lambda'}_{\mu'\nu'} \neq 0$). This surely is not the case if these affinities are to transform as spelt out in (2), because you can have $\Gamma^\lambda_{\mu\nu} = 0$ and $\Gamma^{\lambda'}_{\mu'\nu'} \neq 0$. Against all that is expected from physical and natural reality as we have come to experience it, this literally means a force has a relative sense of existence where it can be made to come into or out of existence by a mere change of the system of coordinates. If anything, coordinates are no more than a convenient way which we use to uniquely label points in space and this should not, in any way imaginable, have any physical effect whatsoever on the resultant physics thereof.

3 Weyl (1918)'s theory

In the first such attempt to bring gravitation and electromagnetism under one mathematical scheme, in which effort one obviously hopes for a unification of these two forces in the resulting theory, Weyl [2] realised that he could forge such a scheme if he were to supplement the metric $g_{\mu\nu}$ of Riemann geometry with a scalar function ϕ as follows:

$$\bar{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}\,. \tag{4}$$

The resulting affine connections from this modified Riemann metric (4) are:

$$\overline{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + W^\lambda_{\mu\nu}\,, \tag{5}$$

where:

$$W^\lambda_{\mu\nu} = g^\lambda{}_\mu \partial_\nu\phi + g^\lambda{}_\nu \partial_\mu\phi - g_{\mu\nu}\partial^\lambda\phi\,, \tag{6}$$

is the tensorial Weyl connection which results from Weyl's supplemented scalar function ϕ. Insofar as its transformation between coordinates is concerned, this new tensorial affine connection of the modified Riemann geometry (hereafter, *Weyl Geometry* (WG)) is no different from the affine connection of Riemann geometry as it transforms as follows:

$$\overline{\Gamma}^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\overline{\Gamma}^\lambda_{\mu\nu} + \frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial x^{\mu'}\partial x^{\nu'}}\,. \tag{7}$$

So, from a viewpoint of topology, WG is the same as RG.

Now, if this Weyl scalar is chosen such that:

$$\phi = \kappa_0 \int A_\alpha dx^\alpha, \tag{8}$$

then the tensorial Weyl connection becomes:

$$W^\lambda_{\mu\nu} = \delta^\lambda{}_\mu A_\nu + \delta^\lambda{}_\nu A_\mu - \delta_{\mu\nu} A^\lambda\,, \tag{9}$$

where A_μ, is (here) a (dimensionless) four-vector and the δ's are the usual Kronecker delta functions, and κ_0 is a constant with the dimensions of inverse length and this constant has been introduced for the purposes of dimensional consistency, since we here assume that the four-vector A_μ and the Weyl scalar ϕ are dimensionless physical quantities.

The versatile and agile Weyl [2] was quick to note that this new Christoffel-Weyl affine (5) is invariant under the following rescaling of the metric $g_{\mu\nu}$ and the four vector A_μ:

$$\left.\begin{array}{lcl} g_{\mu\nu} & \longmapsto & e^{2\Phi} g_{\mu\nu} \\ A_\mu & \longmapsto & A_\mu + \kappa_0^{-1}\partial_\mu\Phi \end{array}\right\} \quad \Rightarrow \quad \overline{\Gamma}^\lambda_{\mu\nu} \longmapsto \overline{\Gamma}^\lambda_{\mu\nu}\,, \tag{10}$$

where $\Phi = \Phi(\boldsymbol{r}, t)$ is a well-behaved, arbitrary, smooth, differentiable, integrable and uniform continuous scalar function.

Now, because Maxwell [13]'s electromagnetic theory is invariant under the same gauge transformation which the four-vector A_μ has been subjected to in (10), the great mind of Weyl seized this beautiful golden moment and identified this four-vector A_μ with the electromagnetic four-vector potential. Weyl went on to assume that the resulting theory was a unified field theory of gravitation and Maxwellian electrodynamics. Weyl's hopes were monumentally dashed, first starting with Einstein's lethal critique of the theory. Later, others joined Einstein in their critique and dismissal of Weyl's theory, where they argued that despite its irresistible grandeur and exquisite beauty, Weyl's theory can not possibly describe the measured reality of the order of the present world.

*A geodesic coordinate system is one in which the Christoffel three symbols ($\Gamma^\lambda_{\mu\nu}$) vanish at all points on the given set of coordinates – i.e. $\Gamma^\lambda_{\mu\nu} = 0$. An example is the flat rectangular (x, y, z) system of coordinates. However, when one moves from this (x, y, z) rectangular system of coordinates to, say, the spherical (r, θ, φ), the resulting affine ($\Gamma^{\lambda'}_{\mu'\nu'}$) is not zero – i.e. $\Gamma^{\lambda'}_{\mu'\nu'} \neq 0$.

4 Modified Weyl theory

Now, following for example Einstein [7, 8], Eddington [14] and Schrödinger [9–11], we strongly felt that the idea of tensorial affinities is the only way to solve the aforementioned topological issues with RG and at the same time, we felt that the beautiful introduction of the four-vector into the framework of RG in WG needed to be preserved at all cost. To us, this meant modifying WG in such a manner that tensorial affinities are attained. For this, we imagined the metric of WG being modified such that it is now given by:

$$\bar{g}_{\mu\nu} = e^{2\chi} g_{\mu\nu}, \tag{11}$$

where, unlike in WG, the function: χ is no longer a scalar, but a pseudo-scalar so designed that the resulting affinities of this new geometry are true tensors.

The new metric given in (11) leads to the following affine connection:

$$\overline{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + Q^{\lambda}_{\mu\nu}\,, \tag{12}$$

where:

$$Q^{\lambda}_{\mu\nu} = g^{\lambda}{}_{\mu} \partial_{\nu}\chi + g^{\lambda}{}_{\nu} \partial_{\mu}\chi - g_{\mu\nu}\partial^{\lambda}\chi \tag{13}$$

is a new affine connection that transforms as follows:

$$Q^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} Q^{\lambda}_{\mu\nu} - \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial x^{\mu'} \partial x^{\nu'}}\,. \tag{14}$$

Because of the transformational properties of the new Q-affine as spelt out in (14) above, the resultant affine $\overline{\Gamma}^{\lambda}_{\mu\nu}$ in (12) is a tensor. In order for the Q-affine to transform as desired in (14), the χ-function must transform as follows:

$$\chi' = \chi - \frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\,. \tag{15}$$

Further, in order for the χ-function to transform as desired in (15), this function ought to be defined as follows:

$$\chi = \ln \Omega\,, \tag{16}$$

where the Ω-function transforms as follows:

$$\Omega' = \Omega \, \exp\left(-\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\right). \tag{17}$$

In this way, tensorial affinities are indeed possible.

5 Unified Field Theory

With the nagging topological defect of RG and WG now out of the way, i.e. the problem of non-tensorial affinities, we realised in [1] that Weyl [2]'s idea can be brought back to life. Instead of just supplementing the Riemann metric with the Weyl-scalar, we have to supplement it with both the Weyl-scalar ϕ and the new χ-function as follows:

$$\bar{g}_{\mu\nu} = e^{2(\phi+\chi)} g_{\mu\nu}\,. \tag{18}$$

This leads to the affine of the emergent geometry now being defined as follows:

$$\overline{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + W^{\lambda}_{\mu\nu} + Q^{\lambda}_{\mu\nu}\,. \tag{19}$$

Just like the affine in the previous section defined in (12), this new affine (19) is also a tensor. From this, one can construct a unified field theory of their choice by identifying the Weyl tensor with a field of their choice. Since all our theories are designed in order to model physical and natural reality, the choice one will have to seek is obviously that which can explain physical and natural reality as we experience it and have come to know it. Our work presented in [1] makes a temarious endeavour to that end.

6 General discussion

Without an iota of doubt, we certainly have demonstrated or shown that it is very much possible to attain tensorial affinities by simple redefining Weyl [2]'s scalar so that it is a pseudo-scalar that is, for better or for worse, forced to yield for us the desired tensorial affinities. In closing, we certainly must hasten to say that our foisting of this pseudo-scalar to yield the desired tensorial affinities has been done well within the permissible and legal confines, domains and provinces of physics, mathematics and philosophy.

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References

1. Nyambuya G. G. Fundamental Geometrodynamic Justification of Gravitomagnetism (I). *Progress in Physics*, 2220, v. 16, 73–91.
2. Weyl H. K. H. Gravitation und Elektrizität. *Sitzungsber. Preuss. Akad. Wiss.*, 1918, v. 26, 465–478.
3. Milgrom M. MOND – A Pedagogical Review. *Acta Phys. Pol. B*, 2001, v. 32, 371–389. arXiv: astro-ph/0112069.
4. Milgrom M. A Modification of the Newtonian Dynamics – Implications for Galaxies. *Ap.J.*, 1983, v. 459, 371–389.
5. Milgrom M. A Modification of the Newtonian Dynamics – Implications for Galaxy Systems. *Ap.J.*, 1983, v. 270, 384–389.
6. Einstein A. Die Feldgleichungun der Gravitation. *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, 1915, 844–847.
7. Einstein A. A Generalisation of the Relativistic Theory of Gravitation. *Ann. Math.*, 1945, v. 46, 578–584.
8. Einstein A. and Straus E. G. *Ann. Math.*, 1946, v. 47, 731. See also: Einstein A. *Rev. Mod. Phys.*, 1948, v. 20, 35; Einstein A. *Can. J. Math.*, 1950. v. 2, 120.
9. Schrödinger E. The Final Affine Field Laws I. *Proceedings of the Royal Irish Academy. A: Math. and Phys. Sc.*, 1945, v. 51, 163–171.
10. Schrödinger E. The Final Affine Field Laws. II. *Proceedings of the Royal Irish Academy. A: Math. and Phys. Sc.*, 1945, v. 51, 205–216.
11. Schrödinger E. The Final Affine Field Laws. III. *Proceedings of the Royal Irish Academy. A: Math. and Phys. Sc.*, 1948, v. 52, 1–9.
12. Einstein A. 1907. Translated in: Schwartz H. M. *Am. J. Phys.*, 1977, v. 45, 10.
13. Maxwell J. C. A Dynamical Theory of the Electromagnetic Field. *Phil. Trans. Royal Soc.*, 1865, v. 155, 459–512.
14. Eddington A. S. The Mathematical Theory of Relativity. Cambridge University Press, Cambridge, 1924.