Physical Observables in General Relativity and the Zelmanov Chronometric Invariants

Dmitri Rabounski and Larissa Borissova
Puschino, Moscow Region, Russia
E-mail: rabounski@yahoo.com, lborissova@yahoo.com

Chronomeric invariants are mathematically determined as the projections of four-dimensional tensorial quantities onto the three-dimensional spatial section and the line of time belonging to a real particular observer. Such projections are physical observables to the observer; it is these quantities that are measurable in his real laboratory and depend on the physical and geometric properties of his local physical space. In other words, chonometric invariants are physical observable quantities in the space-time of General Relativity. Chronometric invariants and the mathematical appararus for their calculation were introduced in 1944 by Abraham L. Zelmanov. In this article, we have collected everything (or almost everything) that we know about chronometric invariants to provide a convenient and most detailed reference to this mathematical apparatus originally scattered throughout many publications.

Physical observables were mathematically determined and introduced into General Relativity in 1941–1944 by Abraham L. Zelmanov (1913–1987), who called them chronometrically invariant quantities or, in brief, *chronometric invariants*. Zelmanov first presented his mathematical apparatus for calculating physical observables in 1944, in the form of his PhD thesis [1]. Later, in 1956–1957, he published a brief review of his theory in two journal articles [2, 3], of which his 1957 presentation is the most useful and complete. A more detailed account of Zelmanov's mathematical apparatus can be found in the respective chapters of our three research monographs [4–6] and in one of our recent journal publications [7].

Chronomerically invariant quantities are determined as the projections of four-dimensional tensorial quantities onto the three-dimensional spatial section and the line of time in the real physical reference frame belonging to a particular observer. Such quantities depend on the physical and geometric properties of his local physical space (his physical reference space) and can be measured in his laboratory. In other words, chonometric invariants are physical observable quantities in the space-time of General Relativity.

For this reason and since we have always sought to obtain a theoretical result that can be registered in laboratory measurements, we used Zelmanov's mathematical apparatus in our research studies. The chronological list of our publications in English and French, wherein we used chronometric invariants, is given in the end of this article.

Unfortunately, it just so happened that after Zelmanov's death in 1987, we remain the only ones in the world who professionally master this mathematical apparatus and apply it in scientific research. In addition, Zelmanov's mathematical apparatus was fragmentarily scattered throughout the aforementioned publications. Some of them pretended to be more or less complete, but were also limited due to the omission of some important parts (not relevant to the specific problem).

For this reason, and also because the problem of physical observables in General Relativity is of great importance for experiment, Pierre A. Millette, Editor of *Progress in Physics*, prompted us to write a compendium containing "everything we know about chronometric invariants and would like to say". Such an article, despite the obvious repetitions with the previous ones, would contain the entire mathematical apparatus of chronometric invariants, which is very convenient for ourselves and our future followers.

We are grateful to Pierre A. Millette for his proposal and will implement it here in this article.

Usually, when doing a research study on General Relativity, we present all equations and their terms in the general covariant (four-dimensional) form. This form has its own advantage as well as a substantial drawback. The advantage is the invariance of general covariant equations and their terms in all transitions from one reference frame to another. The drawback is that they do not show actual three-dimensional quantities, which can be measured in experiments by a real observer in his real physical laboratory. In other words, general covariant equations do not give us *physical observable quantities*, but only an intermediate theoretical result, which is not applicable in practice. Therefore, in order to obtain a theoretical result applicable in practice, we need to formulate our equations in terms of physical observables — the quantities that are experimentally measurable and depend on the physical and geometric properties of the physical local reference space belonging to a real particular observer.

Meanwhile, to determine physical observable quantities in the space-time of General Relativity is not a trivial problem. For instance, a four-dimensional vector, i.e., a contravariant tensor of the 1st rank, has just 4 components: 1 time component and 3 spatial components. In this case, we can heuristically assume that its three spatial components form a three-dimensional observable vector, while its time component is the observable potential of the vector field (which, generally speaking, does not prove that these quantities can actually be observed). A tensor of the 2nd rank, e.g., a rotation or deformation tensor, has 16 components: 1 time component, 9 spatial components and 6 mixed (time-spatial) components. Are the mixed components physical observables? This is another question that seemingly has no definite answer. Tensors of higher ranks have even more components. For instance, the Riemann-Christoffel curvature tensor is a tensor of the 4th rank. It has 256 components. In such a case the problem of the heuristic recognition of physically observable components becomes far more complicated, or even impossible. Besides that, there is an obstacle related to the recognition of observable components of covariant tensors (in which indices occupy the lower position) and of mixed type tensors, which have both lower and upper indices.

Therefore, the most reasonable way out of the labyrinth of heuristic guesses is to create a strict mathematical theory that allows us to calculate observable components for any tensor quantity. As mentioned in the beginning of this article, such a complete mathematical theory was created in 1941–1944 by Zelmanov. His theory was called the mathematical apparatus of physical observable quantities in General Relativity, or, in brief, the *theory of chronometric invariants*.

It should be noted that in the 1930's and 1950's, independently from Zelmanov, some other researchers tried to give a mathematical definition to physical observable quantities in the space-time of General Relativity. In 1939, L. D. Landau and E. M. Lifshitz in their famous *The Classical Theory of Fields* [8] introduced observable time and observable three-dimensional interval similar to Zelmanov's definitions. But, Landau and Lifshitz limited themselves only to this particular case and they did not arrive at general mathematical methods to calculate physical observable quantities in the four-dimensional space-time. In the 1950's, the idea of presenting physical observables in the form of the projections of four-dimensional tensorial quantities onto the three-dimensional spatial section and the time line belonging to an observer was also voiced by the Italian mathematician Carlo Cattaneo [9–12]. Cattaneo highly appreciated Zelmanov's theory of chronometric invariants, and referred to it in his last publication [12]. Nevertheless, when evaluating the scientific contribution of Cattaneo, we must take two facts into account. Firstly, his research was done only in 1958, i.e. 14 years later than Zelmanov. And secondly, his result was very far from a complete theory: he limited himself to general considerations on this problem and did not take into account the physical and geometric observable properties of the local physical space belonging to an observer (as Zelmanov did). Therefore, the projections of four-dimensional tensor quantities considered by Cattaneo do not depend on the observable properties of the observer's reference space and cannot be considered physical observables.

We therefore call physical observable quantities in the space-time of General Relativity the *Zelmanov chronometric invariants* in order to fix this term and Zelmanov's priority in the history of science.

It is also necessary to understand that Zelmanov's mathematical apparatus of chronometric invariants is not just one of many other mathematical techniques used in the General Theory of Relativity, which require an experimental verification of their applicability in practice. The Zelmanov chronometric invariants are physical observables by definition, and there is no other mathematical technique to determine physical observables in General Relativity. In this sense, the mathematical apparatus of chronometric invariants does not require experimental verification, since all quantities that we register in experiments and astronomical observations are chronometric invariants by definition. This fact should always be taken into account, when a researcher seeks to obtain a theoretical result that can be verified in a laboratory experiment or astronomical observations.

Below we present the mathematical apparatus of Zelmanov's chronometric invariants in its entirety, based on his original publications, our personal conversations with him, as well as our own works. So, let us begin.

In order to recognize which of the components of a four-dimensional quantity are physical observables, we consider a physical frame of reference belonging to a real observer, which includes a three-dimensional *coordinate grid* spanned over his *reference body* (a real physical body near him, such as the planet Earth for an Earth-bound observer), at each point of which a *real physical clock* is installed. His reference body, like any other real physical body, has a gravitational field, can rotate and deform, thereby making the local reference space of the observer inhomogeneous and anisotropic. In fact, the reference body and its reference space can be considered as a set of the real physical standards to which the observer compares the results of his measurements. Mathematically, this means that the physical observable quantities registered by an observer are the projections of four-dimensional quantities onto the three-dimensional space (coordinate grid) and the time line of his reference body.

From a geometric point of view, the three-dimensional space of an observer is a *three-dimensional spatial section* drawn in space-time at the time coordinate $x^0 = ct = const$ determined by the moment of observation t. In fact, at any point in space-time, a local spatial section (local space) can be drawn orthogonally to the line of time. If there exists an enveloping curve to such local spatial sections (local three-dimensional spaces) in space-time, these local spatial sections create a global spatial section, everywhere orthogonal to the lines of time that "pierce" it. Such a space is known as a *holonomic space*. If there is not an enveloping curve for such local spaces, then there are only spatial sections locally orthogonal to the lines of time: such a space is *non-holonomic*.

Assume that an observer is at rest with respect to his physical references (his reference body). The reference frame of such an observer always accompanies his reference body in any of its displacements, so such a system is called an *accompanying reference frame*. Any coordinate grid that is at rest with respect to its reference body is connected to another coordinate grid through the transformation

$$\left.\begin{aligned} &\tilde{x}^0 = \tilde{x}^0(x^0, x^1, x^2, x^3) \\ &\tilde{x}^i = \tilde{x}^i(x^1, x^2, x^3), \qquad \frac{\partial \tilde{x}^i}{\partial x^0} = 0 \end{aligned}\right\},$$

where the latter equation means that spatial coordinates in the tilde-marked grid are independent of time in the non-tilded coordinate grid, which is the same as setting a coordinate grid of fixed time lines $x^i = const$ at any point of the grid. Transformation of spatial coordinates is nothing but only transition from one coordinate grid to another within the same spatial section. Transformation of time means changing the whole set of clocks, so this is transition to another spatial section (another three-dimensional reference space). In practice, this means replacing one reference body and all its physical references with another one that has its own physical references. But when using different physical references, the observer will obtain different results of measurement (other observable quantities). Therefore, all physical observable quantities in the reference frame accompanying an observer must be invariant with respect to transformations of time throughout his entire three-dimensional spatial section $x^i = const$. In other words, such quantities must have the property of *chronometric invariance*. That is, all physical observable quantities in the reference frame accompanying an observer are "chronometrically" invariant quantities or, in brief, *chronometric invariants*.

Since the aforementioned transformations of time determine a set of fixed time lines "piercing" the observer's three-dimensional spatial section, chronometric invariants (physical observable quantities) are all those quantities that are invariant with respect to these transformations.

In practice, in order to obtain physical observable quantities in the physical reference frame that accompanies a real observer, we need to calculate chronometrically invariant projections of four-dimensional quantities onto the spatial section and the time line of the observer's physical reference body, and then formulate the projections with chronometrically invariant (physically observable) properties of his local physical reference space.

Therefore, Zelmanov had introduced projection operators that completely characterize the reference space of a particular observer.

The operator of projection onto the time line of an observer is the unit vector of the observer's four-dimensional velocity b^α with respect to his reference body

$$b^\alpha = \frac{dx^\alpha}{ds},$$

which is tangential to his four-dimensional (space-time) trajectory at each of its points. Because any individual reference frame is characterized by its own tangential unit vector b^α, Zelmanov referred to the b^α as the *monad vector*. It is easy to see that since the vector b^α is tangential to the observer's four-dimensional trajectory at each of its points, this vector has unit length

$$b_\alpha b^\alpha = g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = \frac{g_{\alpha\beta}\,dx^\alpha dx^\beta}{ds^2} = +1\,.$$

The operator of projection onto the three-dimensional reference space of the observer (which is an instant spatial section of space-time at the moment of observation) is a four-dimensional symmetric tensor $h_{\alpha\beta}$ having the form

$$h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta\,,$$
$$h^{\alpha\beta} = -g^{\alpha\beta} + b^\alpha b^\beta,$$
$$h^\beta_\alpha = -g^\beta_\alpha + b_\alpha b^\beta.$$

It is easy to see that the vector b^α and the tensor $h_{\alpha\beta}$ have all the necessary properties characteristic of projection operators, namely — the properties

$$b_\alpha b^\alpha = +1\,, \qquad h^\beta_\alpha b^\alpha = 0\,,$$

where the second property follows from the fact that the vector b^α and the tensor $h_{\alpha\beta}$ are orthogonal to each other in space-time: mathematically this means that their common contraction is zero

$$h_{\alpha\beta}b^\alpha = -g_{\alpha\beta}b^\alpha + b_\alpha b^\alpha b_\beta = 0\,,$$
$$h^{\alpha\beta}b_\alpha = -g^{\alpha\beta}b_\alpha + b^\beta b_\alpha b^\alpha = 0\,,$$
$$h^\alpha_\beta b_\alpha = -g^\alpha_\beta b_\alpha + b_\beta b^\alpha b_\alpha = 0\,,$$
$$h^\beta_\alpha b^\alpha = -g^\beta_\alpha b^\alpha + b^\beta b_\alpha b^\alpha = 0\,.$$

In the reference frame accompanying the observer, his three-dimensional velocity with respect to his reference body is zero, which means that $b^i = 0$. As a result, the components of the b^α in the accompanying reference frame are

$$b^0 = \frac{1}{\sqrt{g_{00}}}\,, \qquad b_0 = g_{0\alpha}b^\alpha = \sqrt{g_{00}}\,,$$
$$b^i = 0\,, \qquad b_i = g_{i\alpha}b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}}\,.$$

Therefore, the components of the projection operator $h_{\alpha\beta}$ in the accompanying reference frame ($b^i = 0$) are

$$h_{00} = 0\,, \qquad h^{00} = -g^{00} + \frac{1}{g_{00}}\,, \qquad h^0_0 = 0\,,$$
$$h_{0i} = 0\,, \qquad h^{0i} = -g^{0i}, \qquad h^i_0 = \delta^i_0 = 0\,,$$
$$h_{i0} = 0\,, \qquad h^{i0} = -g^{i0}, \qquad h^0_i = \frac{g_{i0}}{g_{00}}\,,$$
$$h_{ik} = -g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}}\,, \quad h^{ik} = -g^{ik}, \quad h^i_k = -g^i_k = \delta^i_k\,.$$

The projection of a tensor onto the time line of an observer is the result of its contraction with the monad vector b^α of his reference frame.

The projection of a tensor onto the three-dimensional spatial section of the observer (his three-dimensional reference space) is the result of its contraction with the tensor $h_{\alpha\beta}$ of his reference frame.

Despite the fact that such projections of a tensor of the 1st rank (a vector) are chronometric invariants, i.e., physical observables, not all such projections (contractions) of higher rank tensors have the property of chronometric invariance. To solve this problem, Zelmanov developed a general mathematical method for calculating chronometrically invariant (physically observable) projections of any four-dimensional general covariant tensor and formulated it as a theorem. We refer to it as *Zelmanov's theorem*.

Zelmanov's theorem: Let there be a four-dimensional tensor $Q^{\mu\nu\dots\rho}_{\alpha\beta\dots\sigma}$ of the r-th rank, where $Q^{ik\dots p}_{00\dots0}$ is the three-dimensional part of $Q^{\mu\nu\dots\rho}_{00\dots0}$, in which all upper indices are non-zero, and all m lower indices are zeroes. Then,

$$T^{ik\dots p} = (g_{00})^{-\frac{m}{2}}\, Q^{ik\dots p}_{00\dots0}$$

is a chronometrically invariant three-dimensional contravariant tensor of the $(r-m)$-th rank. This means that the chr.inv.-tensor $T^{ik\dots p}$ is the result of m-fold projection of the initial tensor $Q^{\mu\nu\dots\rho}_{\alpha\beta\dots\sigma}$ onto the time line by the indices $\alpha,\beta\dots\sigma$ and onto the spatial section by $r-m$ indices $\mu,\nu\dots\rho$.

According to this theorem, the chronometrically invariant (physically observable) projections of a four-dimensional vector Q^α are the quantities

$$b^\alpha Q_\alpha = \frac{Q_0}{\sqrt{g_{00}}}\,, \qquad h^i_\alpha Q^\alpha = q^i,$$

while the chr.inv.-projections of a symmetric tensor of the 2nd rank $Q^{\alpha\beta}$ are the quantities

$$b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}\,, \quad h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}\,, \quad h^i_\alpha h^k_\beta\, Q^{\alpha\beta} = Q^{ik},$$

where, in the case of an antisymmetric tensor of the 2nd rank, the first chr.inv.-projection is zero, because $Q_{00} = Q^{00} = 0$ for any antisymmetric 2nd rank tensor.

The chr.inv.-projections of a four-dimensional coordinate interval dx^α are the physically observable time interval

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\, dx^i,$$

and the interval of the physically observable coordinates dx^i, which are the same as the regular spatial coordinates. Thus, the three-dimensional chr.inv.-vector

$$\mathrm{v}^i = \frac{dx^i}{d\tau}\,, \qquad \mathrm{v}_i \mathrm{v}^i = h_{ik}\, \mathrm{v}^i \mathrm{v}^k = \mathrm{v}^2$$

is the physically observable velocity of a particle, which is different from the particle's coordinate velocity

$$u^i = \frac{dx^i}{dt}\,.$$

At isotropic trajectories (trajectories of light), the v^i transforms into the three-dimensional chr.inv.-vector of the physically observable velocity of light

$$c^i = \mathrm{v}^i = \frac{dx^i}{d\tau}\,, \quad c_i c^i = h_{ik}\, c^i c^k = c^2.$$

When we project the fundamental metric tensor $g_{\alpha\beta}$ onto the three-dimensional spatial section of an observer (which is his three-dimensional reference space)

$$h^i_\alpha h^k_\beta\, g^{\alpha\beta} = g^{ik} = -h^{ik}, \quad h^\alpha_i h^\beta_k g_{\alpha\beta} = g_{ik} - b_i b_k = -h_{ik}\,,$$

we see that the three-dimensional part of the projection operator $h_{\alpha\beta}$, i.e., the three-dimensional tensor h_{ik}, the components of which have the form

$$h_{ik} = -g_{ik} + b_i b_k\,, \quad h^{ik} = -g^{ik}, \quad h^i_k = -g^i_k = \delta^i_k\,,$$

is the *chr.inv.-metric tensor* or, in other words, the metric tensor physically observed in the reference frame accompanying the observer.

The chr.inv.-metric tensor h_{ik} has all properties of the fundamental metric tensor $g_{\alpha\beta}$ throughout the observer's three-dimensional spatial section (his three-dimensional reference space), i.e., it satisfies the condition

$$h^i_\alpha h^\alpha_k = \delta^i_k - b_k b^i = \delta^i_k\,, \qquad \delta^i_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where δ^i_k is the unit three-dimensional tensor. The tensor δ^i_k is the three-dimensional part of the four-dimensional unit tensor δ^α_β, which can be used to lift and lower indices in four-dimensional quantities. For this reason, the chr.inv.-metric tensor h_{ik} can lift and lower indices in chronometrically invariant quantities.

Using $g_{\alpha\beta}$ from $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$, we obtain the four-dimensional interval $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ in the form

$$ds^2 = b_\alpha b_\beta\, dx^\alpha dx^\beta - h_{\alpha\beta}\, dx^\alpha dx^\beta$$

expressed with the projection operators b^α and $h_{\alpha\beta}$. Because $b_\alpha dx^\alpha = c\,d\tau$, the first term of the above formula transforms into $b_\alpha b_\beta\, dx^\alpha dx^\beta = c^2 d\tau^2$. The second term of this formula, $h_{\alpha\beta}\, dx^\alpha dx^\beta = d\sigma^2$, in the reference frame accompanying the observer is the square of the three-dimensional physically observable interval

$$d\sigma^2 = h_{ik}\, dx^i dx^k,$$

since $h_{\alpha\beta}$ has all properties of the fundamental metric tensor $g_{\alpha\beta}$ in the accompanying reference frame.

As a result, the four-dimensional interval written in terms of physically observable chr.inv.-quantities has the form

$$ds^2 = c^2 d\tau^2 - d\sigma^2.$$

Obviously, the physical observables (chr.inv.-projections of four-dimensional quantities) registered by an observer depend on the physical and geometric observable properties of the observer's local space (his physical reference space), with which, therefore, all chr.inv.-quantities and equations must be expressed. Therefore, Zelmanov deduced the basic observable properties of the reference space accompanying an observer and introduced them into the theory.

Two main physical observable properties of the accompanying reference space can be obtained using the chr.inv.-derivation operators with respect to time and the spatial coordinates. The mentioned chr.inv.-derivation operators introduced by Zelmanov have the form

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^0}$$

and are non-commutative, so the difference between the 2nd derivatives is not zero

$$\frac{{}^*\partial^2}{\partial x^i \partial t} - \frac{{}^*\partial^2}{\partial t\,\partial x^i} = \frac{1}{c^2} F_i \frac{{}^*\partial}{\partial t},$$

$$\frac{{}^*\partial^2}{\partial x^i \partial x^k} - \frac{{}^*\partial^2}{\partial x^k \partial x^i} = \frac{2}{c^2} A_{ik} \frac{{}^*\partial}{\partial t}.$$

Here, A_{ik} is the three-dimensional antisymmetric chr.inv.-tensor of the angular velocity with which the reference space of the observer rotates

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right),$$

where v_i is the linear velocity of this rotation

$$v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}}, \qquad v^i = -c\,g^{0i}\sqrt{g_{00}},$$

$$v_i = h_{ik} v^k, \qquad v^2 = v_k v^k = h_{ik} v^i v^k.$$

In addition, the v_i gives detailed formulae for the physically observable time interval $d\tau$ and the chr.inv.-metric tensor h_{ik}, which are

$$d\tau = \sqrt{g_{00}}\,dt - \frac{1}{c^2} v_i dx^i, \quad h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k.$$

The quantity F_i is the three-dimensional chr.inv.-vector of the gravitational inertial force

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right) = \frac{1}{1-\frac{\mathrm{w}}{c^2}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right),$$

where

$$\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\,\right)$$

is the gravitational potential, the origin of which is the gravitational field of the observer's reference body. In the framework of quasi-Newtonian approximation, i.e., in a weak gravitational field at velocities much lower than the velocity of light and in the absence of rotation of the space, the F_i transforms into the non-relativistic gravitational force

$$F_i = \frac{\partial \mathrm{w}}{\partial x^i}.$$

It should be noted that the quantities w and v_i do not have the property of chronometric invariance, despite the fact that $v_i = h_{ik} v^k$ is obtained as for a chr.inv.-quantity, through lowering the upper index by the chr.inv.-metric tensor h_{ik}. On the other hand, the vector of the gravitational inertial force F_i and the tensor of the angular velocity of rotation of the observer's space, A_{ik}, built using them, are chr.inv.-quantities.

The chr.inv.-quantities F_i and A_{ik} are related to each other by two identities, which we call the *Zelmanov identities*

$$\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0,$$

$$\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\left(F_i A_{km} + F_k A_{mi} + F_m A_{ik}\right) = 0.$$

In addition to rotation and the presence of a gravitational field, the real reference body of an observer can deform. In this case, the observer's reference space with its coordinate grid deforms accordingly, which must be taken into account in experiments. Mathematically, this factor manifests itself in the non-stationarity of the physically observable chr.inv.-metric h_{ik} of the observer's space and must be taken into account in the physically observable chr.inv.-quantities registered by him. For this reason, Zelmanov had introduced the three-dimensional symmetric chr.inv.-tensor D_{ik} characterizing the rate of deformations of the observer's space

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}, \qquad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t},$$

$$D = h^{ik} D_{ik} = D^n_n = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}, \qquad h = \det\|h_{ik}\|.$$

Zelmanov had also introduced a theorem linking the holonomity of space-time to the tensor of the angular velocity of rotation of the observer's three-dimensional space.

> Zelmanov's theorem on the holonomity of space-time: The identical equality to zero of the tensor A_{ik} in a four-dimensional region of space-time is the necessary and sufficient condition for the orthogonality of the spatial sections to the time lines everywhere in this region.

In other words, $A_{ik} \neq 0$ in a non-holonomic space-time region, and $A_{ik} = 0$ in a holonomic one. Naturally, if the three-dimensional spatial sections are everywhere orthogonal to the time lines (in such a case the space-time region is holonomic), all the quantities g_{0i} are equal to zero. Since $g_{0i} = 0$, we have

$v_i = 0$ and $A_{ik} = 0$ too. Therefore, we also refer to the tensor A_{ik} as the *space non-holonomity tensor*.

The space-time of Special Relativity (Minkowski space) in the Galilean reference frame, as well as some cases of the space-time in General Relativity, do not rotate ($A_{ik} = 0$). These are examples of holonomic spaces: time lines are orthogonal to spatial sections in them. Rotating spaces ($A_{ik} \neq 0$) are non-holonomic; time lines are non-orthogonal to three-dimensional spatial sections in such spaces.

To understand why the rotation of a three-dimensional spatial section of space-time makes this spatial section non-orthogonal to the time lines "piercing" it, consider a *locally geodesic reference frame*. Within the infinitesimal vicinity of any point in such a reference frame, the fundamental metric tensor has the form

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{\mu\nu}}{\partial \tilde{x}^\rho \partial \tilde{x}^\sigma}\right)(\tilde{x}^\rho - x^\rho)(\tilde{x}^\sigma - x^\sigma) + \dots \,,$$

which means that the numerical values of its components in the infinitesimal vicinity of any point differ from those at this point itself only in the 2nd order terms and the higher other terms, which can be neglected. Therefore, at any point in a locally geodesic reference frame, the fundamental metric tensor (within the 2nd order terms withheld) is constant, while the 1st derivatives of the metric tensor, i.e., the Christoffel symbols, are zeroes.

It is obvious that in any Riemannian space within the infinitesimal vicinity of any point of the space a locally geodesic reference frame can be set up. As a result, at any point belonging to the locally geodesic reference frame, a flat space can be set up tangential to the Riemannian space so that the locally geodesic reference frame in the Riemannian space is a globally geodesic frame in the tangential flat space. Since the fundamental metric tensor is constant in the flat space, there in the infinitesimal vicinity of any point in the Riemannian space the quantities $\tilde{g}_{\mu\nu}$ converge to those of the tensor $g_{\mu\nu}$ in the tangential flat space. This means that, in the tangential flat space, we can set up a system of the basis vectors $\vec{e}_{(\alpha)}$ tangential to the curved coordinate lines of the Riemannian space. Because the coordinate lines of a Riemannian space are curved (in a general case), and, in the case where the space is non-holonomic, are not even orthogonal to each other, the lengths of the basis vectors are sometimes substantially different from unit length.

Consider the world-vector $d\vec{r}$ of an infinitesimal displacement from such a point, i.e., $d\vec{r} = \{dx^0, dx^1, dx^2, dx^3\}$. Then $d\vec{r} = \vec{e}_{(\alpha)} dx^\alpha$, where its components $e_{(\alpha)}$ are

$$\vec{e}_{(0)} = \left\{e^0_{(0)}, 0, 0, 0\right\}, \qquad \vec{e}_{(1)} = \left\{0, e^1_{(1)}, 0, 0\right\},$$

$$\vec{e}_{(2)} = \left\{0, 0, e^2_{(2)}, 0\right\}, \qquad \vec{e}_{(3)} = \left\{0, 0, 0, e^3_{(3)}\right\}.$$

The scalar product of the vector $d\vec{r}$ with itself is equal to $d\vec{r}d\vec{r} = ds^2$. On the other hand, it is $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. Thus, we obtain the general formula

$$g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)}e_{(\beta)} \cos(x^\alpha; x^\beta).$$

According to this formula we have

$$g_{00} = e^2_{(0)} \,,$$

while, on the other hand, $\sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}$. Hence, the length $e_{(0)}$ of the time basis vector $\vec{e}_{(0)}$ tangential to the time line $x^0 = ct$ is expressed with the gravitational potential w as

$$e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,.$$

The stronger the gravitational potential w, the smaller $e_{(0)}$ is than 1. In the case of gravitational collapse ($\mathrm{w} = c^2$), the length of the time basis vector $\vec{e}_{(0)}$ becomes zero: $e_{(0)} = 0$.

Thus, according to the above general formula, the component g_{0i} is expressed as

$$g_{0i} = e_{(0)}e_{(i)} \cos(x^0; x^i),$$

while, according to the definition of v_i, we have

$$g_{0i} = -\frac{1}{c}\, v_i \left(1 - \frac{\mathrm{w}}{c^2}\right) = -\frac{1}{c}\, v_i\, e_{(0)}\,,$$

whence we obtain the formula for v_i, which takes into account the angle of inclination of the time lines to the three-dimensional spatial section of space-time, i.e.

$$v_i = -c\, e_{(i)} \cos(x^0; x^i).$$

In addition, since the above general formula gives

$$g_{ik} = e_{(i)}e_{(k)} \cos(x^i; x^k),$$

and according to the definition of the chr.inv.-metric tensor h_{ik} (page 7), we obtain the formula for h_{ik}, which also takes into account the angle of inclination of the time lines to the three-dimensional spatial section

$$h_{ik} = e_{(i)}e_{(k)} \Big[\cos(x^0; x^i)\cos(x^0; x^k) - \cos(x^i; x^k)\Big].$$

From the above formula for v_i, we see that from a geometric point of view, the linear velocity v_i with which the three-dimensional reference space of an observer rotates is the projection (scalar product) of the time basis vector $\vec{e}_{(0)}$ of his reference space onto the spatial basis vectors $\vec{e}_{(i)}$, multiplied by the velocity of light. If the spatial sections of a space (space-time) are everywhere orthogonal to the time lines thereby giving the space holonomity, then $\cos(x^0; x^i) = 0$ and, hence, $v_i = 0$. In a non-holonomic space, the spatial sections are not orthogonal to the lines of time: $\cos(x^0; x^i) \neq 0$.

Generally $|\cos(x^0; x^i)| \leqslant 1$, hence the linear velocity v_i with which the three-dimensional reference space of an observer rotates cannot exceed the velocity of light.

If somewhere the conditions $F_i = 0$ and $A_{ik} = 0$ are met in common, there the conditions $g_{00} = 1$ and $g_{0i} = 0$ are present as well (the conditions $g_{00} = 1$ and $g_{0i} = 0$ can be satisfied through the transformation of time). In such a region, according to the definition of $d\tau$ (page 6), we have $d\tau = dt$: so, the difference between the coordinate time t and the physically observable time τ disappears in the absence of gravitational fields and rotation of space. In other words, according to the theory of chronometric invariants, the difference between the coordinate time t and the physically observable time τ comes from both gravitation and rotation attributed to the local reference space of the observer (in fact — from his reference body, which is a real physical body near him, for example, the planet Earth for an Earth-bound observer), or from each of the mentioned two factors separately.

On the other hand, it is doubtful to find such a region of the Universe where gravitational fields or rotation of the background space are clearly absent. Therefore, in practice the physically observable time τ differs from the coordinate time t. This means that the real space of our Universe is non-holonomic: it rotates and is filled with gravitational fields, while a holonomic space, free from rotation and gravity, can only be a local approximation to it.

The condition of holonomity of a space (space-time) is directly linked to the problem of integrability of time in it. In a non-holonomic space, the formula for the physically observable time interval $d\tau$ has no integrating multiplier, i.e., it cannot be transformed to the form

$$d\tau = A\,dt\,,$$

where the multiplier A depends on only t and x^i. In this case the formula for $d\tau$ (page 6) has a non-zero second term depending on the coordinate interval dx^i and g_{0i}. On the contrary, in a holonomic space, we have $A_{ik} = 0$, so $g_{0i} = 0$. In this case, the second term of the formula for $d\tau$ is zero, while the first term is the coordinate time interval dt with an integrating multiplier

$$A = \sqrt{g_{00}} = f(x^0, x^i)\,,$$

so we can write the integral

$$d\tau = \int \sqrt{g_{00}}\,dt\,.$$

Hence time is integrable in a holonomic space ($A_{ik} = 0$), while it cannot be integrated if the space is non-holonomic ($A_{ik} \neq 0$). In the case where time is integrable, i.e., in a holonomic space, we can synchronize the clocks installed at two distantly located points by moving a control clock along the path between these two points. In the case where time cannot be integrated (in a non-holonomic space), synchronization of clocks in two distant points is impossible in principle: the larger is the distance between these two points, the more is the deviation of time on these clocks.

The space of our planet Earth, is non-holonomic due to the daily rotation of it around the Earth's axis. Hence, two clocks installed at different points on the surface of the Earth should manifest a deviation between the intervals of time registered on each of them. The larger is the distance between these clocks, the larger is the deviation of the physically observable time expected to be registered on them. This effect was surely verified by the well-known Hafele-Keating experiment performed in October 1971 by Joseph C. Hafele together with Richard E. Keating [13–15] and then successfully repeated by the UK's National Measurement Laboratory commonly with the BBC on its 25th anniversary in 2005 [16]. This experiment concerned with displacing standard atomic clocks by a jet airplane around the terrestrial globe, where rotation of the Earth's space sensibly changed the measured time. During the flight along the Earth's rotation, the local space of an observer on board of the airplane had more rotation than the space of another observer who stayed fixed on the airfield. During the flight against the Earth's rotation it was vice versa. The atomic clocks on board the airplane showed a significant deviation of the observed time depending on the velocity of rotation of the observer's space.

Since synchronization of clocks at various points on the Earth's surface is the highly important task of metrology, marine navigation, aviation, and orbital space flights, corrections for desynchronization were introduced in early times in the form of tables of empirically obtained corrections that take the Earth's rotation into account. Now, thanks to the theory of chronometric invariants, we know the origin of the corrections and therefore can calculate them on the basis of General Relativity.

With Zelmanov's definitions of chr.inv.-quantities above, we can not only calculate the physically observable chr.inv.-projections of any four-dimensional general covariant quantity or equation of theoretical physics, but also express them in terms of the physically observable chr.inv.-properties F^i, A_{ik}, and D_{ik} characteristic of the local reference space of a particular observer.

The Christoffel symbols (coherence coefficients of space) appear in the absolute derivatives, the equations of motion, and somewhere else in the equations of theoretical physics. The Christoffel symbols are not tensors [17]. Nevertheless, they can be expressed in terms of physical observable quantities. Following the analogy with the regular Christoffel symbols of the 2nd rank $\Gamma^{\alpha}_{\mu\nu}$ and the regular Christoffel symbols of the 1st rank $\Gamma_{\mu\nu,\sigma}$

$$\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\sigma}\,\Gamma_{\mu\nu,\sigma} = \frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right),$$

Zelmanov had introduced the chr.inv.-Christoffel symbols of the 2nd rank and 1st rank

$$\Delta^{i}_{jk} = h^{im}\Delta_{jk,m} = \frac{1}{2}\,h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^{k}} + \frac{{}^*\partial h_{km}}{\partial x^{j}} - \frac{{}^*\partial h_{jk}}{\partial x^{m}}\right),$$

where the only difference is that the chr.inv.-Christoffel symbols use the chr.inv.-metric tensor h_{ik} instead of the fundamental metric tensor $g_{\alpha\beta}$.

It is not a problem to find out how the regular Christoffel symbols are expressed in terms of the physically observable chr.inv.-properties characteristic of the reference space of an observer. Expressing the components of $g^{\alpha\beta}$ and then the 1st derivatives of $g_{\alpha\beta}$ with F_i, A_{ik}, D_{ik}, w, and v_i, after some algebra we obtain

$$\Gamma_{00,0} = -\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial t}\,,$$

$$\Gamma_{00,i} = \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F_i + \frac{1}{c^4}\,v_i\,\frac{\partial \mathrm{w}}{\partial t}\,,$$

$$\Gamma_{0i,0} = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^i}\,,$$

$$\Gamma_{0i,j} = -\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_{ij}+A_{ij}+\frac{1}{c^2}\,F_j v_i\right)+\frac{1}{c^3}\,v_j\,\frac{\partial \mathrm{w}}{\partial x^i}\,,$$

$$\Gamma_{ij,0} = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[D_{ij}-\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)+\frac{1}{2c^2}(F_i v_j+F_j v_i)\right],$$

$$\Gamma_{ij,k} = -\Delta_{ij,k}+\frac{1}{c^2}\left[v_i A_{jk}+v_j A_{ik}+\frac{1}{2}\,v_k\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)- -\frac{1}{2c^2}\,v_k\,(F_i v_j+F_j v_i)\right]+\frac{1}{c^4}\,F_k v_i v_j\,,$$

$$\Gamma^0_{00} = -\frac{1}{c^3}\left[\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial t}+\left(1-\frac{\mathrm{w}}{c^2}\right)v_k F^k\right],$$

$$\Gamma^k_{00} = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F^k,$$

$$\Gamma^0_{0i} = \frac{1}{c^2}\left[-\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial x^i}+v_k\left(D^k_i+A^{\cdot k}_{i\cdot}+\frac{1}{c^2}\,v_i F^k\right)\right],$$

$$\Gamma^k_{0i} = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D^k_i+A^{\cdot k}_{i\cdot}+\frac{1}{c^2}\,v_i F^k\right),$$

$$\Gamma^0_{ij} = -\frac{1}{c\,(1-\frac{\mathrm{w}}{c^2})}\left\{-D_{ij}+\frac{1}{c^2}\,v_n\times \times\left[v_j(D^n_i+A^{\cdot n}_{i\cdot})+v_i(D^n_j+A^{\cdot n}_{j\cdot})+\frac{1}{c^2}\,v_i v_j F^n\right]+ +\frac{1}{2}\left(\frac{\partial v_i}{\partial x^j}+\frac{\partial v_j}{\partial x^i}\right)-\frac{1}{2c^2}(F_i v_j+F_j v_i)-\Delta^n_{ij}\,v_n\right\},$$

$$\Gamma^k_{ij} = \Delta^k_{ij}-\frac{1}{c^2}\left[v_i\left(D^k_j+A^{\cdot k}_{j\cdot}\right)+v_j\left(D^k_i+A^{\cdot k}_{i\cdot}\right)+\frac{1}{c^2}\,v_i v_j F^k\right].$$

Respectively, some components of the regular Christoffel symbols are linked to the chr.inv.-properties of the observer's space by the following relations

$$D^i_k+A^{\cdot i}_{k\cdot} = \frac{c}{\sqrt{g_{00}}}\left(\Gamma^i_{0k}-\frac{g_{0k}\Gamma^i_{00}}{g_{00}}\right),$$

$$F^k = -\frac{c^2\,\Gamma^k_{00}}{g_{00}}\,,$$

$$g^{i\alpha}g^{k\beta}\,\Gamma^m_{\alpha\beta} = h^{iq}h^{ks}\Delta^m_{qs}\,.$$

By analogy with the respective absolute derivatives, Zelmanov had also introduced the chr.inv.-derivatives

$${}^*\nabla_i\,Q_k = \frac{{}^*\partial Q_k}{dx^i}-\Delta^l_{ik}\,Q_l\,,$$

$${}^*\nabla_i\,Q^k = \frac{{}^*\partial Q^k}{dx^i}+\Delta^k_{il}\,Q^l,$$

$${}^*\nabla_i\,Q_{jk} = \frac{{}^*\partial Q_{jk}}{dx^i}-\Delta^l_{ij}\,Q_{lk}-\Delta^l_{ik}\,Q_{jl}\,,$$

$${}^*\nabla_i\,Q^k_j = \frac{{}^*\partial Q^k_j}{dx^i}-\Delta^l_{ij}\,Q^k_l+\Delta^k_{il}\,Q^l_j\,,$$

$${}^*\nabla_i\,Q^{jk} = \frac{{}^*\partial Q^{jk}}{dx^i}+\Delta^j_{il}\,Q^{lk}+\Delta^k_{il}\,Q^{jl},$$

$${}^*\nabla_i\,Q^i = \frac{{}^*\partial Q^i}{\partial x^i}+\Delta^j_{ji}\,Q^i, \qquad \Delta^j_{ji} = \frac{{}^*\partial\ln\sqrt{h}}{\partial x^i}\,,$$

$${}^*\nabla_i\,Q^{ji} = \frac{{}^*\partial Q^{ji}}{\partial x^i}+\Delta^j_{il}\,Q^{il}+\Delta^l_{li}\,Q^{ji}, \quad \Delta^l_{li} = \frac{{}^*\partial\ln\sqrt{h}}{\partial x^i}\,.$$

In particular, they show the following properties of the chr.inv.-metric tensor h_{ik}

$${}^*\nabla_i\,h_{jk} = 0\,, \quad {}^*\nabla_i\,h^k_j = 0\,, \quad {}^*\nabla_i\,h^{jk} = 0\,.$$

Next we give an account of tensor calculus in terms of physical observables (chronometric invariants).

Assume that there is a space (not necessarily metric) in which there is an arbitrary reference frame $\{x^\alpha\}$. Let this space contain an object G determined by n functions f_n of the x^α coordinates. Let us know the transformation rule to calculate these n functions in any other reference frame $\{\tilde{x}^\alpha\}$ in this space. If the n functions f_n and also the transformation rule have been given in the space, then G is a *geometric object*, which in the system $\{x^\alpha\}$ has axial components $f_n\,(x^\alpha)$, while in any other system $\{\tilde{x}^\alpha\}$ it has components $\tilde{f}_n\,(\tilde{x}^\alpha)$.

Assume that a tensor object (tensor) of zero rank is any geometric object φ, transformable according to the rule

$$\tilde{\varphi} = \varphi\,\frac{\partial x^\alpha}{\partial\tilde{x}^\alpha}\,,$$

where the index takes numbers of all coordinate axes one-by-one (this notation is also known as *by-component notation* or *tensor notation*). Any tensor of zero rank has a single component and is called *scalar*.

From a geometric point of view, any scalar is a point to which a certain number is attributed. Therefore, a scalar field is a set of points of the space, which have a common property. For instance, a point mass is a scalar, while a distributed mass (a gas, for instance) makes up a scalar field.

It should be noted that the algebraic notations for a tensor and a tensor field are the same. The field of a tensor in a space is represented as the tensor in a given point of the space, but its presence in other points everywhere in this region of the space is assumed.

Contravariant tensors of the 1st rank A^{α} are geometric objects with components, transformable according to the rule

$$\tilde{A}^{\alpha} = A^{\mu}\,\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\mu}}\,.$$

From a geometric point of view, such an object is an n-dimensional vector. For instance, the vector of an elementary displacement dx^{α} is a contravariant tensor of the 1st rank.

Contravariant tensors of the 2nd rank $A^{\alpha\beta}$ are geometric objects transformable according to the rule

$$\tilde{A}^{\alpha\beta} = A^{\mu\nu}\,\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\mu}}\frac{\partial\tilde{x}^{\beta}}{\partial x^{\nu}}\,.$$

From a geometric point of view, such an object is the area (parallelogram) formed by two vectors. For this reason, contravariant tensors of the 2nd rank are also called *bivectors*.

So forth, contravariant tensors of higher ranks are formulated as the following geometric objects

$$\tilde{A}^{\alpha\dots\sigma} = A^{\mu\dots\tau}\,\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\mu}}\cdots\frac{\partial\tilde{x}^{\sigma}}{\partial x^{\tau}}\,.$$

A vector field or a higher rank tensor field are space distributions of the respective tensor quantities. For instance, because a mechanical strength characterizes both its own magnitude and direction, its distribution in a physical body can be presented by a vector field.

Covariant tensors of the 1st rank A_{α} are geometric objects, transformable according to the rule

$$\tilde{A}_{\alpha} = A_{\mu}\,\frac{\partial x^{\mu}}{\partial\tilde{x}^{\alpha}}\,.$$

Thus, the gradient of a scalar field φ, i.e., the quantity

$$A_{\alpha} = \frac{\partial\varphi}{\partial x^{\alpha}}\,,$$

is a covariant tensor of the 1st rank. This is because for a regular invariant we have $\tilde{\varphi} = \varphi$, then

$$\frac{\partial\tilde{\varphi}}{\partial\tilde{x}^{\alpha}} = \frac{\partial\tilde{\varphi}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial\tilde{x}^{\alpha}} = \frac{\partial\varphi}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial\tilde{x}^{\alpha}}\,.$$

Covariant tensors of the 2nd rank $A_{\alpha\beta}$ are geometric objects with the transformation rule

$$\tilde{A}_{\alpha\beta} = A_{\mu\nu}\,\frac{\partial x^{\mu}}{\partial\tilde{x}^{\alpha}}\frac{\partial x^{\nu}}{\partial\tilde{x}^{\beta}}\,,$$

hence, covariant tensors of higher ranks are formulated as

$$\tilde{A}_{\alpha\dots\sigma} = A_{\mu\dots\tau}\,\frac{\partial x^{\mu}}{\partial\tilde{x}^{\alpha}}\cdots\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}}\,.$$

Mixed tensors are tensors of the 2nd rank or of higher ranks with both upper and lower indices. For instance, any mixed symmetric tensor A^{α}_{β} is a geometric object, transformable according to the rule

$$\tilde{A}^{\alpha}_{\beta} = A^{\mu}_{\nu}\,\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial\tilde{x}^{\beta}}\,.$$

Tensor objects exist both in metric and non-metric spaces. In non-metric spaces, as it is known, the distance between any two points cannot be measured. This is in contrast to metric spaces. In the theories of space-time-matter, such as the General Theory of Relativity and its extensions, metric spaces are taken under consideration. This is because the core of such theories is the measurement of time intervals and spatial lengths, that is nonsense in a non-metric space.

Any tensor has a^n components, where a is its dimension and n is the rank. For instance, a four-dimensional tensor of zero rank has 1 component, a tensor of the 1st rank has 4 components, a tensor of the 2nd rank has 16 components, a tensor of the 4th rank (for example, the Riemann-Christoffel curvature tensor) has 256 components, and so on.

Indices in a geometric object, marking its axial components, are found not in tensors only, but in other geometric objects as well. For this reason, if we encounter a quantity in component notation, it is not necessarily a tensor quantity.

In practice, to know whether a given object is a tensor or not, we need to know a formula for this object in a reference frame and to transform it to any other reference frame. For instance, consider the classic question: are Christoffel's symbols (i.e., the coherence coefficients of space) tensors? To answer this question, we need to calculate the Christoffel symbols in a tilde-marked reference frame

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \tilde{g}^{\alpha\sigma}\,\widetilde{\Gamma}_{\mu\nu,\sigma}\,, \quad \widetilde{\Gamma}_{\mu\nu,\sigma} = \frac{1}{2}\left(\frac{\partial\tilde{g}_{\mu\sigma}}{\partial\tilde{x}^{\nu}} + \frac{\partial\tilde{g}_{\nu\sigma}}{\partial\tilde{x}^{\mu}} - \frac{\partial\tilde{g}_{\mu\nu}}{\partial\tilde{x}^{\sigma}}\right)$$

proceeding from the general formula of them in a non-marked reference frame.

First, we calculate the terms in the brackets. The fundamental metric tensor like any other covariant tensor of the 2nd rank, is transformable to the tilde-marked reference frame according to the following rule

$$\tilde{g}_{\mu\sigma} = g_{\varepsilon\tau}\,\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}}\,.$$

Because the quantity $g_{\varepsilon\tau}$ depends on the non-tilde-marked coordinates, its derivative with respect to the tilde-marked coordinates (which are functions of the non-tilded ones) is calculated according to the rule

$$\frac{\partial g_{\varepsilon\tau}}{\partial\tilde{x}^{\nu}} = \frac{\partial g_{\varepsilon\tau}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}}\,,$$

and thus the first term in the brackets, taking the rule of transformation of the fundamental metric tensor into account, takes the form

$$\frac{\partial\tilde{g}_{\mu\sigma}}{\partial\tilde{x}^{\nu}} = \frac{\partial g_{\varepsilon\tau}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}}\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}} + g_{\varepsilon\tau}\left(\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}}\frac{\partial^2 x^{\varepsilon}}{\partial\tilde{x}^{\nu}\partial\tilde{x}^{\mu}} + \frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial^2 x^{\tau}}{\partial\tilde{x}^{\nu}\partial\tilde{x}^{\sigma}}\right).$$

Calculating the rest of the terms of the tilde-marked Christoffel symbols and transposing their free indices we obtain

$$\widetilde{\Gamma}_{\mu\nu,\sigma} = \Gamma_{\varepsilon\rho,\tau}\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}} + g_{\varepsilon\tau}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}}\frac{\partial^2 x^{\varepsilon}}{\partial\tilde{x}^{\mu}\partial\tilde{x}^{\nu}},$$

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\gamma}_{\varepsilon\rho}\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\gamma}}\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}} + \frac{\partial\tilde{x}^{\alpha}}{\partial x^{\gamma}}\frac{\partial^2 x^{\gamma}}{\partial\tilde{x}^{\mu}\partial\tilde{x}^{\nu}}.$$

We see that the Christoffel symbols are not transformed in the same way as tensors, hence they are not tensors.

Tensors can be represented as matrices. But in practice, such a form can only be possible for tensors of the 1st or 2nd rank (one-row and flat matrices, respectively). For instance, the tensor of an elementary four-dimensional displacement can be represented in the form of a one-row matrix

$$dx^{\alpha} = (dx^0,\, dx^1,\, dx^2,\, dx^3),$$

the four-dimensional fundamental metric tensor can be represented in the form of a flat matrix

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix},$$

and tensors of the 3rd rank are three-dimensional matrices. Representing tensors of higher ranks as matrices is problematic and not visual.

Now let us turn to tensor algebra, the branch of tensor calculus that focuses on algebraic operations with tensors.

Only same-type tensors of the same rank with indices in the same position can be added or subtracted. Adding up two n-rank tensors of the same type gives a new tensor of the same type and rank, the components of which are the sums of the corresponding components of the added tensors. For instance

$$A^{\alpha} + B^{\alpha} = D^{\alpha}, \qquad A^{\alpha}_{\beta} + B^{\alpha}_{\beta} = D^{\alpha}_{\beta}.$$

Multiplication is allowed not only for tensors of the same type, but also for any tensors of any rank. External multiplication of a tensor of the n-rank by a tensor of the m-rank gives a new tensor of the $(n+m)$-rank

$$A_{\alpha\beta}B_{\gamma} = D_{\alpha\beta\gamma}, \qquad A_{\alpha}B^{\beta\gamma} = D^{\beta\gamma}_{\alpha}.$$

Contraction is the multiplication of tensors of the same rank when some of their indices are the same. Contraction of tensors across all indices yields a scalar

$$A_{\alpha}B^{\alpha} = C, \qquad A^{\gamma}_{\alpha\beta}B^{\alpha\beta}_{\gamma} = D.$$

Often the multiplication of tensors entails the contraction of some of their indices. Such multiplication is known as inner multiplication, which means that some indices become contracted when the tensors are multiplied. Below is an example of internal multiplication

$$A_{\alpha\sigma}B^{\sigma} = D_{\alpha}, \qquad A^{\gamma}_{\alpha\sigma}B^{\beta\sigma}_{\gamma} = D^{\beta}_{\alpha}.$$

Using internal multiplication of geometric objects we can determine whether they are tensors or not. This is the so-called *fraction theorem*.

> FRACTION THEOREM: If $B^{\sigma\beta}$ is a tensor and its internal multiplication with a geometric object $A(\alpha,\sigma)$ is a tensor $D(\alpha,\beta)$, i.e., $A(\alpha,\sigma)\,B^{\sigma\beta} = D(\alpha,\beta)$, then this object $A(\alpha,\sigma)$ is also a tensor.

According to this theorem, if internal multiplication of an object $A_{\alpha\sigma}$ with a tensor $B^{\sigma\beta}$ gives another tensor D^{β}_{α}

$$A_{\alpha\sigma}B^{\sigma\beta} = D^{\beta}_{\alpha},$$

then this object $A_{\alpha\sigma}$ is a tensor. Or, if internal multiplication of an object A^{α}_{σ} and a tensor $B^{\sigma\beta}$ gives a tensor $D^{\alpha\beta}$

$$A^{\alpha\cdot}_{\cdot\sigma}B^{\sigma\beta} = D^{\alpha\beta},$$

then the object $A^{\alpha\cdot}_{\cdot\sigma}$ is a tensor.

The geometric properties of any metric space are determined by its fundamental metric tensor, which can lift and lower the indices in the objects of this metric space. In Riemannian spaces, the space metric has a square form, which is $ds^2 = g_{\alpha\beta}\,dx^{\alpha}dx^{\beta}$ and is known also as the *Riemannian metric*, so the fundamental metric tensor of a Riemannian space is a tensor of the 2nd rank $g_{\alpha\beta}$. The mixed fundamental metric tensor g^{β}_{α} is equal to the unit tensor $g^{\beta}_{\alpha} = g_{\alpha\sigma}\,g^{\sigma\beta} = \delta^{\beta}_{\alpha}$. The diagonal components of the unit tensor are units, while its rest (non-diagonal) components are zeroes. Using the unit tensor we can replace the indices in four-dimensional quantities

$$\delta^{\beta}_{\alpha}A_{\beta} = A_{\alpha}, \qquad \delta^{\nu}_{\mu}\delta^{\sigma}_{\rho}A^{\mu\rho} = A^{\nu\sigma}.$$

Contracting any tensor of the 2nd rank with the fundamental metric tensor $g_{\alpha\beta}$ yields a scalar known as the tensor spur or its trace

$$g^{\alpha\beta}A_{\alpha\beta} = A^{\sigma}_{\sigma} = A.$$

For example, the spur of the fundamental metric tensor in a four-dimensional pseudo-Riemannian space is 4

$$g_{\alpha\beta}\,g^{\alpha\beta} = g^{\sigma}_{\sigma} = g^0_0 + g^1_1 + g^2_2 + g^3_3 = \delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3 = 4.$$

As mentioned on page 6, the chr.inv.-metric tensor h_{ik} has all properties of the fundamental metric tensor $g_{\alpha\beta}$ throughout the observer's three-dimensional spatial section (his three-dimensional reference space). Therefore, h_{ik} can lower, lift and replace indices in chr.inv.-quantities. Accordingly, the spur (trace) of any three-dimensional chr.inv.-tensor is obtained by contracting it with h_{ik}. For instance, the spur (trace) of the tensor of the rate of deformations of the observer's space, D_{ik}, is the chr.inv.-scalar

$$D = D_m^m = h^{ik} D_{ik} \,,$$

the physical sense of which is the rate of relative expansion or contraction of the elementary volume of the observer's reference space.

The scalar product of two vectors A^α and B^α (tensors of the 1st rank) in a four-dimensional pseudo-Riemannian space is formulated as

$$g_{\alpha\beta} A^\alpha B^\beta = A_\alpha B^\alpha = A_0 B^0 + A_i B^i .$$

Scalar product is the result of contraction, because the multiplication of vectors contracts all their indices. Therefore, the scalar product of two vectors (tensors of the 1st rank) is always a scalar (tensor of zero rank). If both the vectors are the same, their scalar product

$$g_{\alpha\beta} A^\alpha A^\beta = A_\alpha A^\alpha = A_0 A^0 + A_i A^i$$

is the square of the given vector A^α, the length of which is expressed as

$$A = |A^\alpha| = \sqrt{g_{\alpha\beta} A^\alpha A^\beta} \,.$$

The four-dimensional pseudo-Riemannian space, which is the space-time of General Relativity, by its definition has the sign-alternating metric, i.e., the fundamental metric tensor has the sign-alternating signature $(+---)$ or $(-+++)$. In this case, the lengths of four-dimensional vectors can be real, imaginary or zero. Vectors with non-zero (real or imaginary) lengths are known as *non-isotropic vectors*; they are tangential to non-isotropic trajectories. Vectors with zero length are known as *isotropic vectors*; they are tangential to isotropic trajectories (trajectories of light-like particles).

In the three-dimensional Euclidean space, the scalar product of two vectors is a scalar quantity, the numerical value of which is equal to the product of their lengths and the cosine of the angle between them

$$A_i B^i = |A^i| |B^i| \cos (A^i ; B^i) \,.$$

From the above formula it follows that the scalar product of two vectors is zero, if the vectors are orthogonal to each other. In other words, from a geometric point of view, the scalar product of two vectors is the projection of one vector onto the other. If the vectors are the same, then the vector is projected onto itself, so the result of this projection is the square of its length.

Theoretically, at each point of any Riemannian space, a tangential flat space can be set, the basis vectors of which are tangential to the basis vectors of the Riemannian space at this point. Then, the metric of the tangential flat space is also the metric of the Riemannian space at this point. Therefore, the above formula is also true, if we consider the angle between the three-dimensional coordinate lines and the time lines in the space thereby replacing the Roman (three-dimensional spatial) indices with the Greek (four-dimensional) ones.

Denote the chr.inv.-projections of arbitrary vectors A^α and B^α onto the time line and the three-dimensional spatial section of an observer as follows

$$a = \frac{A_0}{\sqrt{g_{00}}} \,, \qquad a^i = A^i ,$$

$$b = \frac{B_0}{\sqrt{g_{00}}} \,, \qquad b^i = B^i ,$$

then their remaining components have the form

$$A^0 = \frac{a + \frac{1}{c} v_i a^i}{1 - \frac{w}{c^2}} \,, \quad A_i = -a_i - \frac{a}{c} v_i \,,$$

$$B^0 = \frac{b + \frac{1}{c} v_i b^i}{1 - \frac{w}{c^2}} \,, \quad B_i = -b_i - \frac{b}{c} v_i \,.$$

Substituting the chr.inv.-projections of the vectors A^α and B^α into the formulae for $A_\alpha B^\alpha$ and $A_\alpha A^\alpha$, we obtain

$$A_\alpha B^\alpha = ab - a_i b^i = ab - h_{ik} a^i b^k ,$$

$$A_\alpha A^\alpha = a^2 - a_i a^i = a^2 - h_{ik} a^i a^k .$$

From here, we see that the square of the length of any vector is the difference between the squares of the lengths of its time and spatial chr.inv.-projections. If both these projections are the same, then the vector's length is zero, so the vector is isotropic. Hence, any isotropic vector equally belongs to the time line and the spatial section. The equality of its time projection to its spatial projection also means that this vector is orthogonal to itself. If its time projection is "longer" than its spatial projection, then this vector is real. If the spatial projection is "longer", then this vector is imaginary.

The latter can be illustrated by the square of the length of the space-time interval

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = dx_\alpha dx^\alpha = dx_0 dx^0 + dx_i dx^i ,$$

which in terms of chr.inv.-quantities has the form

$$ds^2 = c^2 d\tau^2 - dx_i dx^i = c^2 d\tau^2 - h_{ik} dx^i dx^k = c^2 d\tau^2 - d\sigma^2 .$$

Its length ds can be real, imaginary or zero, depending on whether ds is time-like $c^2 d\tau^2 > d\sigma^2$, which is the case along sublight-speed real trajectories, space-like $c^2 d\tau^2 < d\sigma^2$, which is the case of imaginary superluminal-speed trajectories, or isotropic $c^2 d\tau^2 = d\sigma^2$, which is the case of light-like (isotropic) trajectories, respectively.

The vector product of two vectors A^α and B^α is a tensor of the 2nd rank $V^{\alpha\beta}$, obtained from their external multiplication according to the rule

$$V^{\alpha\beta} = [A^\alpha ; B^\beta] = \frac{1}{2} (A^\alpha B^\beta - A^\beta B^\alpha) = \frac{1}{2} \begin{vmatrix} A^\alpha & A^\beta \\ B^\alpha & B^\beta \end{vmatrix} .$$

As it is easy to see, in this case the order in which the vectors are multiplied matters, i.e., the order in which we write down the tensor indices is important. For this reason, the tensors obtained as vector products are called *antisymmetric tensors*. In an antisymmetric tensor we have $V^{\alpha\beta} = -V^{\beta\alpha}$, where its indices being moved "reserve" their places as dots, $g_{\alpha\sigma}V^{\sigma\beta} = V_{\alpha\cdot}^{\cdot\beta}$, thereby showing the place from where the specific index was moved. In symmetric tensors there is no need to "reserve" places for moved indices, because the order in which they appear does not matter. For example, the fundamental metric tensor is symmetric $g_{\alpha\beta} = g_{\beta\alpha}$, and the Riemann-Christoffel tensor of the curvature of space $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ is symmetric with respect to transposition over a pair of its indices and antisymmetric within each pair of the indices. It is obvious that only tensors of the 2nd rank or higher ranks can be symmetric or antisymmetric.

All diagonal components of any antisymmetric tensor by its definition are zeroes. For instance, in an antisymmetric tensor of the 2nd rank we have

$$V^{\alpha\alpha} = [A^{\alpha}; B^{\alpha}] = \frac{1}{2}\left(A^{\alpha}B^{\alpha} - A^{\alpha}B^{\alpha}\right) = 0\,.$$

In the three-dimensional Euclidean space, the numerical value of the vector product of two vectors is defined as the area of the parallelogram formed by them and is equal to the product of their moduli multiplied by the sine of the angle between them

$$V^{ik} = |A^i||B^k|\sin\left(A^i; B^k\right).$$

This means that the vector product of two vectors, i.e., any antisymmetric tensor of the 2nd rank, is a pad oriented in space according to the directions of the vectors forming it.

The contraction of an antisymmetric tensor $V_{\alpha\beta}$ with any symmetric tensor $A^{\alpha\beta} = A^{\alpha}A^{\beta}$ is zero. Naturally, since $V_{\alpha\alpha} = 0$ and $V_{\alpha\beta} = -V_{\beta\alpha}$ we have

$$V_{\alpha\beta}A^{\alpha}A^{\beta} = V_{00}A^0A^0 + V_{0i}A^0A^i + V_{i0}A^iA^0 + V_{ik}A^iA^k = 0\,.$$

According to the theory of chronometric invariants, an antisymmetric tensor of the 2nd rank $V^{\alpha\beta}$ has the following chr.inv.-projections

$$\frac{V_{00}}{g_{00}} = 0\,,$$

$$\frac{V_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = -\frac{V_{\cdot 0}^{i\cdot}}{\sqrt{g_{00}}} = \frac{1}{2}\left(ab^i - ba^i\right),$$

$$V^{ik} = \frac{1}{2}\left(a^ib^k - a^kb^i\right),$$

which are expressed here with the chr.inv.-projections of its forming (multiplied) vectors A^{α} and B^{α}: here a and b are the chr.inv.-projections of the multiplied vectors A^{α} and B^{α} onto the time line of the observer, and a^i and b^i are their chr.inv.-projections onto the observer's spatial section (which is his three-dimensional reference space).

The first chr.inv.-projection of the antisymmetric tensor $V^{\alpha\beta}$ is zero, since in any antisymmetric tensor all its diagonal components are zeroes. The third physically observable chr.inv.-quantity V^{ik} is the projection of the tensor $V^{\alpha\beta}$ onto the observer's spatial section. It is analogous to a vector product in the three-dimensional space. The second chr.inv.-quantity of the above is the space-time (mixed) projection of $V^{\alpha\beta}$. It has no equivalent among the components of a regular three-dimensional vector product.

The square of an antisymmetric tensor of the 2nd rank $V^{\alpha\beta}$, formulated with the chr.inv.-projections of its forming vectors A^{α} and B^{α}, is calculated as

$$V_{\alpha\beta}V^{\alpha\beta} = \frac{1}{2}\left(a_ia^ib_kb^k - a_ib^ia_kb^k\right) + \\ + ab\,a_ib^i - \frac{1}{2}\left(a^2b_ib^i - b^2a_ia^i\right).$$

The asymmetry of tensor fields is determined by reference antisymmetric tensors. Such references in the Galilean reference frame* are Levi-Civita's tensors: for four-dimensional quantities this is the four-dimensional completely antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$, while for three-dimensional quantities this is the three-dimensional completely antisymmetric unit tensor e^{ikm}. The components of the Levi-Civita tensors, which have all indices different, are either +1 or −1 depending on the number of transpositions of their indices. All the remaining components, i.e., those having at least two coinciding indices, are zeroes. Moreover, with the space signature (+−−−) we are using, all non-zero contravariant components of the Levi-Civita tensors have the opposite sign to their corresponding covariant components†. For instance, in the Minkowski space we have

$$g_{\alpha\sigma}g_{\beta\rho}g_{\mu\tau}g_{\nu\gamma}e^{\sigma\rho\tau\gamma} = g_{00}g_{11}g_{22}g_{33}e^{0123} = -e^{0123},$$

$$g_{i\alpha}g_{k\beta}g_{m\gamma}e^{\alpha\beta\gamma} = g_{11}g_{22}g_{33}e^{123} = -e^{123},$$

since $g_{00} = 1$ and $g_{11} = g_{22} = g_{33} = -1$ with the space signature (+−−−) we are using. In this case, the components of the tensor $e^{\alpha\beta\mu\nu}$ are

$$e^{0123} = +1, \quad e^{1023} = -1, \quad e^{1203} = +1, \quad e^{1230} = -1,$$

$$e_{0123} = -1, \quad e_{1023} = +1, \quad e_{1203} = -1, \quad e_{1230} = +1,$$

and the components of the tensor e^{ikm} are

$$e^{123} = +1, \quad e^{213} = -1, \quad e^{231} = +1,$$

$$e_{123} = -1, \quad e_{213} = +1, \quad e_{231} = -1.$$

*A Galilean reference frame is one that does not rotate, is not subject to deformation, and falls freely in the space-time of Special Relativity (Minkowski space). The time lines in the Galilean reference frame are linear, as are the three-dimensional coordinate axes.

†If the space signature is (−+++), then what has been said is true only for the four-dimensional Levi-Civita tensor $e^{\alpha\beta\mu\nu}$. The components of the three-dimensional Levi-Civita tensor e^{ikm} will have the same sign as well as the corresponding components of the e_{ikm} tensor.

In general, the tensor $e^{\alpha\beta\mu\nu}$ is related to the tensor e^{ikm} as follows $e^{0ikm} = e^{ikm}$. Because we have an arbitrary choice for the sign of the first component, we can choose $e^{0123} = -1$ and $e^{123} = -1$. Then the remaining components of e^{ikm} will change respectively.

Multiplying the four-dimensional antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$ by itself we obtain a regular tensor of the 8th rank with the non-zero components determined by the matrix

$$e^{\alpha\beta\mu\nu} e_{\sigma\tau\rho\gamma} = -\begin{pmatrix} \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} & \delta^{\alpha}_{\rho} & \delta^{\alpha}_{\gamma} \\ \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} & \delta^{\beta}_{\rho} & \delta^{\beta}_{\gamma} \\ \delta^{\mu}_{\sigma} & \delta^{\mu}_{\tau} & \delta^{\mu}_{\rho} & \delta^{\mu}_{\gamma} \\ \delta^{\nu}_{\sigma} & \delta^{\nu}_{\tau} & \delta^{\nu}_{\rho} & \delta^{\nu}_{\gamma} \end{pmatrix}.$$

The remaining properties of the tensor $e^{\alpha\beta\mu\nu}$ are deduced from the previous by means of contraction of their indices

$$e^{\alpha\beta\mu\nu} e_{\sigma\tau\rho\nu} = -\begin{pmatrix} \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} & \delta^{\alpha}_{\rho} \\ \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} & \delta^{\beta}_{\rho} \\ \delta^{\mu}_{\sigma} & \delta^{\mu}_{\tau} & \delta^{\mu}_{\rho} \end{pmatrix},$$

$$e^{\alpha\beta\mu\nu} e_{\sigma\tau\mu\nu} = -2\begin{pmatrix} \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} \\ \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} \end{pmatrix} = -2\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \delta^{\beta}_{\sigma}\delta^{\alpha}_{\tau}\right),$$

$$e^{\alpha\beta\mu\nu} e_{\sigma\beta\mu\nu} = -6\,\delta^{\alpha}_{\sigma}, \quad e^{\alpha\beta\mu\nu} e_{\alpha\beta\mu\nu} = -6\,\delta^{\alpha}_{\alpha} = -24.$$

Multiplying the three-dimensional antisymmetric unit tensor e^{ikm} by itself we obtain a regular tensor of the 6th rank

$$e^{ikm} e_{rst} = \begin{vmatrix} \delta^{i}_{r} & \delta^{i}_{s} & \delta^{i}_{t} \\ \delta^{k}_{r} & \delta^{k}_{s} & \delta^{k}_{t} \\ \delta^{m}_{r} & \delta^{m}_{s} & \delta^{m}_{t} \end{vmatrix}.$$

The remaining properties of the tensor e^{ikm} are

$$e^{ikm} e_{rsm} = -\begin{pmatrix} \delta^{i}_{r} & \delta^{i}_{s} \\ \delta^{k}_{r} & \delta^{k}_{s} \end{pmatrix} = \delta^{i}_{s}\delta^{k}_{r} - \delta^{i}_{r}\delta^{k}_{s},$$

$$e^{ikm} e_{rkm} = 2\,\delta^{i}_{r}, \quad e^{ikm} e_{ikm} = 2\,\delta^{i}_{i} = 6.$$

The completely antisymmetric unit tensor determines for a tensor object its corresponding *pseudotensor*, marked with asterisk. For instance, any four-dimensional scalar, vector and tensors of the 2nd, 3rd, and 4th ranks have corresponding four-dimensional pseudotensors of the following ranks

$$V^{*\alpha\beta\mu\nu} = e^{\alpha\beta\mu\nu} V, \quad V^{*\alpha\beta\mu} = e^{\alpha\beta\mu\nu} V_{\nu}, \quad V^{*\alpha\beta} = \frac{1}{2}\, e^{\alpha\beta\mu\nu} V_{\mu\nu},$$

$$V^{*\alpha} = \frac{1}{6}\, e^{\alpha\beta\mu\nu} V_{\beta\mu\nu}, \quad V^{*} = \frac{1}{24}\, e^{\alpha\beta\mu\nu} V_{\alpha\beta\mu\nu},$$

where pseudotensors of the 1st rank, such as $V^{*\alpha}$, are called *pseudovectors*, while pseudotensors of zero rank, such as V^{*}, are called *pseudoscalars*. Any tensor and its corresponding pseudotensor are known as *dual* to each other to emphasize their common genesis. So, three-dimensional antisymmetric tensors have their corresponding three-dimensional pseudotensors

$$V^{*ikm} = e^{ikm} V, \qquad V^{*ik} = e^{ikm} V_{m},$$

$$V^{*i} = \frac{1}{2}\, e^{ikm} V_{km}, \quad V^{*} = \frac{1}{6}\, e^{ikm} V_{ikm}.$$

Pseudotensors are called such because, in contrast to regular tensors, they do not change when reflected with respect to one of the coordinate axes. For instance, when reflected with respect to the abscissa axis $x^1 = -\tilde{x}^1$, $x^2 = \tilde{x}^2$, $x^3 = \tilde{x}^3$, the reflected component of an antisymmetric tensor V_{ik}, orthogonal to x^1, is $\widetilde{V}_{23} = -V_{23}$, while the dual component of the pseudovector V^{*i} retains the original sign unchanged

$$V^{*1} = \frac{1}{2}\, e^{1km} V_{km} = \frac{1}{2}\left(e^{123} V_{23} + e^{132} V_{32}\right) = V_{23},$$

$$\widetilde{V}^{*1} = \frac{1}{2}\, \tilde{e}^{1km} \widetilde{V}_{km} = \frac{1}{2}\, e^{k1m} \widetilde{V}_{km} = \frac{1}{2}\left(e^{213} \widetilde{V}_{23} + e^{312} \widetilde{V}_{32}\right) = V_{23}.$$

Since any four-dimensional antisymmetric tensor of the 2nd rank and its dual pseudotensor are of the same rank, their contraction yields a pseudoscalar, which is

$$V_{\alpha\beta} V^{*\alpha\beta} = V_{\alpha\beta}\, e^{\alpha\beta\mu\nu} V_{\mu\nu} = e^{\alpha\beta\mu\nu} B_{\alpha\beta\mu\nu} = B^{*}.$$

The square of a pseudotensor $V^{*\alpha\beta}$ and a pseudovector V^{*i}, expressed with their dual tensors, are

$$V_{*\alpha\beta} V^{*\alpha\beta} = e_{\alpha\beta\mu\nu} V^{\mu\nu} e^{\alpha\beta\rho\sigma} V_{\rho\sigma} = -24\, V_{\mu\nu} V^{\mu\nu},$$

$$V_{*i} V^{*i} = e_{ikm} V^{km} e^{ipq} V_{pq} = 6\, V_{km} V^{km}.$$

We cannot set a Galilean reference frame in an inhomogeneous and anisotropic pseudo-Riemannian space. In such a general space, the antisymmetry references of tensor fields depend on the inhomogeneity and anisotropy of the space itself, which are determined by the fundamental metric tensor, and a reference antisymmetric tensor is the four-dimensional completely antisymmetric discriminant tensor

$$E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}, \qquad E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}.$$

The proof is the following. Transformation of the four-dimensional completely antisymmetric unit tensor $e_{\alpha\beta\mu\nu}$ from a Galilean (non-tilde-marked) reference frame into an arbitrary (tilde-marked) reference frame is

$$\tilde{e}_{\alpha\beta\mu\nu} = \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\alpha}} \frac{\partial x^{\gamma}}{\partial \tilde{x}^{\beta}} \frac{\partial x^{\varepsilon}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\tau}}{\partial \tilde{x}^{\nu}}\, e_{\sigma\gamma\varepsilon\tau} = J e_{\alpha\beta\mu\nu},$$

where

$$J = \det\left\| \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\sigma}} \right\| = \det \begin{Vmatrix} \frac{\partial x^0}{\partial \tilde{x}^0} & \frac{\partial x^0}{\partial \tilde{x}^1} & \frac{\partial x^0}{\partial \tilde{x}^2} & \frac{\partial x^0}{\partial \tilde{x}^3} \\ \frac{\partial x^1}{\partial \tilde{x}^0} & \frac{\partial x^1}{\partial \tilde{x}^1} & \frac{\partial x^1}{\partial \tilde{x}^2} & \frac{\partial x^1}{\partial \tilde{x}^3} \\ \frac{\partial x^2}{\partial \tilde{x}^0} & \frac{\partial x^2}{\partial \tilde{x}^1} & \frac{\partial x^2}{\partial \tilde{x}^2} & \frac{\partial x^2}{\partial \tilde{x}^3} \\ \frac{\partial x^3}{\partial \tilde{x}^0} & \frac{\partial x^3}{\partial \tilde{x}^1} & \frac{\partial x^3}{\partial \tilde{x}^2} & \frac{\partial x^3}{\partial \tilde{x}^3} \end{Vmatrix}$$

is the determinant of Jacobi's matrix known also as the Jacobian of the transformation. Because the fundamental metric tensor $g_{\alpha\beta}$ is transformable according to the rule

$$\tilde{g}_{\alpha\beta} = g_{\mu\nu}\,\frac{\partial x^{\mu}}{\partial\tilde{x}^{\alpha}}\frac{\partial x^{\nu}}{\partial\tilde{x}^{\beta}}$$

and since its determinant in the tilde-marked frame is

$$\tilde{g} = \det\left\| g_{\mu\nu}\,\frac{\partial x^{\mu}}{\partial\tilde{x}^{\alpha}}\frac{\partial x^{\nu}}{\partial\tilde{x}^{\beta}}\right\| = J^2 g\,,$$

then, in the Galilean (non-tilde-marked) reference frame,

$$g = \det\left\|g_{\alpha\beta}\right\| = \det\left\|\begin{matrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{matrix}\right\| = -1,$$

and, hence, $J^2 = -\tilde{g}^2$. Denoting $\tilde{e}_{\alpha\beta\mu\nu}$ in an arbitrary reference frame as $E_{\alpha\beta\mu\nu}$ and writing down the metric tensor in a regular non-tilde-marked form, we obtain

$$E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\,\sqrt{-g}$$

as expected at the very beginning, which was to be proved. In the same way, we obtain the transformation rule

$$E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}$$

for the components $E^{\alpha\beta\mu\nu}$, because for them

$$g = \tilde{g}\,\tilde{J}^2, \qquad \tilde{J} = \det\left\|\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\sigma}}\right\|.$$

The discriminant tensor $E^{\alpha\beta\mu\nu}$ is not a physical observable quantity. For this reason, Zelmanov had introduced the four-dimensional discriminant tensor $\varepsilon^{\alpha\beta\gamma}$

$$\varepsilon^{\alpha\beta\gamma} = h^{\alpha}_{\mu}h^{\beta}_{\nu}h^{\gamma}_{\rho}b_{\sigma}E^{\sigma\mu\nu\rho} = b_{\sigma}E^{\sigma\alpha\beta\gamma},$$

$$\varepsilon_{\alpha\beta\gamma} = h^{\mu}_{\alpha}h^{\nu}_{\beta}h^{\rho}_{\gamma}b^{\sigma}E_{\sigma\mu\nu\rho} = b^{\sigma}E_{\sigma\alpha\beta\gamma}\,,$$

which in the accompanying reference frame of an observer ($b^i = 0$) and taking into account that $\sqrt{-g} = \sqrt{h}\,\sqrt{g_{00}}$ according to the theory of chronometric invariants transforms into the three-dimensional chr.inv.-discriminant tensor ε^{ikm}

$$\varepsilon^{ikm} = b_0 E^{0ikm} = \sqrt{g_{00}}\,E^{0ikm} = \frac{e^{ikm}}{\sqrt{h}}\,,$$

$$\varepsilon_{ikm} = b^0 E_{0ikm} = \frac{E_{0ikm}}{\sqrt{g_{00}}} = e_{ikm}\,\sqrt{h},$$

for which, as is easy to obtain, we have

$$^*\nabla_l\,\varepsilon_{ijk} = 0, \qquad ^*\nabla_l\,\varepsilon^{ijk} = 0,$$

$$\frac{^*\partial\varepsilon_{ijk}}{\partial t} = \varepsilon_{ijk}D, \qquad \frac{^*\partial\varepsilon^{ijk}}{\partial t} = -\varepsilon^{ijk}D,$$

where D is the spur (trace) of the chr.inv.-tensor D_{ik} characterizing the rate of deformations of the observer's space

$$D = h^{ik}D_{ik} = D^n_n = \frac{^*\partial\ln\sqrt{h}}{\partial t}\,, \qquad h = \det\|h_{ik}\|\,.$$

The three-dimensional chr.inv.-discriminant tensor ε^{ikm} is the physical observable reference of the asymmetry of tensor fields in the observer's reference space. Using the ε^{ikm}, we can transform antisymmetric chr.inv.-tensors into the corresponding chr.inv.-pseudotensors.

For example, for the chr.inv.-tensor A_{ik} of the angular velocity of rotation of the observer's space, we have the chr.inv.-pseudovector Ω^{*i} of this rotation

$$\Omega^{*i} = \frac{1}{2}\,\varepsilon^{ikm}A_{km}\,, \quad \Omega_{*i} = \frac{1}{2}\,\varepsilon_{imn}A^{mn}, \quad A^{ik} = \varepsilon^{mik}\,\Omega_{*m}\,,$$

$$\varepsilon^{ipq}\,\Omega_{*i} = \frac{1}{2}\,\varepsilon^{ipq}\varepsilon_{imn}A^{mn} = \frac{1}{2}\left(\delta^p_m\delta^q_n - \delta^q_m\delta^p_n\right)A^{mn} = A^{pq}.$$

With the chr.inv.-pseudovector Ω^{*i} the Zelmanov identities (page 7) connecting the chr.inv.-quantities F_i and A_{ik} take the form, respectively,

$$\frac{2}{\sqrt{h}}\frac{^*\partial}{\partial t}(\sqrt{h}\,\Omega^{*i}) + \varepsilon^{ijk}\,{}^*\nabla_j F_k = 0\,,$$

$$^*\nabla_k\,\Omega^{*k} + \frac{1}{c^2}\,F_k\,\Omega^{*k} = 0\,.$$

Next we consider the absolute differential and absolute directional derivative.

In geometry, a *differential* of a function is its variation between two infinitely close points with the coordinates x^{α} and $x^{\alpha}+dx^{\alpha}$. Respectively, the *absolute differential* in an n-dimensional space represents the change of an n-dimensional quantity between two infinitely close points in this space. For continuous functions, which we commonly deal with in practice, their variations between infinitely close points are infinitesimal. But in order to determine an infinitesimal variation of a tensor quantity, we cannot use a simple "difference" between its numerical values at the neighbouring points x^{α} and $x^{\alpha}+dx^{\alpha}$, because tensor algebra does not determine it. This ratio can only be determined using the rules for transforming tensors from one reference frame to another. As a consequence, differential operators and the results of their application to tensors must be tensors.

For instance, the absolute differential of a tensor quantity is a tensor of the same rank as the original tensor itself. The absolute differential of a scalar φ is the scalar

$$\mathrm{D}\varphi = \frac{\partial\varphi}{\partial x^{\alpha}}\,dx^{\alpha},$$

which in the accompanying reference frame of an observer ($b^i = 0$) takes the form

$$\mathrm{D}\varphi = \frac{^*\partial\varphi}{\partial t}\,d\tau + \frac{^*\partial\varphi}{\partial x^i}\,dx^i,$$

where, apart from the three-dimensional observable differential (second term), there is an additional term that takes into account the dependence of the absolute differential $\mathrm{D}\varphi$ on the physically observable time interval $d\tau$.

The absolute differential of a contravariant vector A^α is formulated with the absolute derivation operator ∇ (nabla) and has the following form

$$\mathrm{D}A^\alpha = \nabla_\sigma A^\alpha dx^\sigma = \frac{\partial A^\alpha}{\partial x^\sigma}\, dx^\sigma + \Gamma^\alpha_{\mu\sigma} A^\mu dx^\sigma = \\ = dA^\alpha + \Gamma^\alpha_{\mu\sigma} A^\mu dx^\sigma,$$

where $\nabla_\sigma A^\alpha$ is the absolute derivative of A^α with respect to x^σ, and d stands for regular differentials

$$\nabla_\sigma A^\alpha = \frac{\partial A^\alpha}{\partial x^\sigma} + \Gamma^\alpha_{\mu\sigma} A^\mu,$$

$$d = \frac{\partial}{\partial x^\alpha}\, dx^\alpha.$$

Formulating the absolute differential with physical observable quantities is equivalent to projecting its general covariant form onto the time line and the spatial section in the accompanying reference frame of an observer. According to the theory of chronometric invariants, the physically observable chr.inv.-projections of the absolute differential of a vector A^α are the quantities

$$T = b_\alpha \mathrm{D}A^\alpha = \frac{g_{0\alpha}\mathrm{D}A^\alpha}{\sqrt{g_{00}}}\,, \qquad B^i = h^i_\alpha \mathrm{D}A^\alpha.$$

Denoting the chr.inv.-projections of the vector A^α as

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = A^i,$$

we calculate its remaining components, which, when expressed in terms of the φ and q^i take the form

$$A_0 = \varphi\left(1 - \frac{\mathrm{w}}{c^2}\right), \quad A^0 = \frac{\varphi + \frac{1}{c}\, v_i q^i}{1 - \frac{\mathrm{w}}{c^2}}\,, \quad A_i = -q_i - \frac{\varphi}{c}\, v_i\,.$$

Taking the chr.inv.-formula for the regular differential

$$d = \frac{^*\partial}{\partial t}\, d\tau + \frac{^*\partial}{\partial x^i}\, dx^i$$

into account, we substitute them and also the regular Christoffel symbols expressed in terms of chr.inv.-quantities (see page 10) into the T and B^i. As a result we obtain the chr.inv.-projections of the absolute differential of the vector A^α in the final chr.inv.-form

$$T = b_\alpha \mathrm{D}A^\alpha = d\varphi + \frac{1}{c}\left(-F_i q^i d\tau + D_{ik} q^i dx^k\right),$$

$$B^i = h^i_\sigma \mathrm{D}A^\sigma = dq^i + \left(\frac{\varphi}{c}\, dx^k + q^k d\tau\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) - \\ - \frac{\varphi}{c}\, F^i d\tau + \Delta^i_{mk}\, q^m dx^k.$$

The *directional derivative* of a function is its change with respect to the elementary displacement along the given direction. The *absolute directional derivative* in an n-dimensional space is the change of an n-dimensional quantity with respect to an elementary n-dimensional interval along the given direction in the space.

For instance, the absolute derivative of a scalar function φ to a direction along a curve $x^\alpha = x^\alpha(\rho)$, where ρ is a non-zero monotone parameter along this curve, expresses the rate at which this function φ changes

$$\frac{\mathrm{D}\varphi}{d\rho} = \frac{d\varphi}{d\rho}\,,$$

which in the accompanying reference frame of an observer is

$$\frac{\mathrm{D}\varphi}{d\rho} = \frac{^*\partial\varphi}{\partial t}\frac{d\tau}{d\rho} + \frac{^*\partial\varphi}{\partial x^i}\frac{dx^i}{d\rho}\,.$$

The absolute derivative of a vector A^α to the given direction tangential to a curve $x^\alpha = x^\alpha(\rho)$ is

$$\frac{\mathrm{D}A^\alpha}{d\rho} = \nabla_\sigma A^\alpha \frac{dx^\sigma}{d\rho} = \frac{dA^\alpha}{d\rho} + \Gamma^\alpha_{\mu\sigma} A^\mu \frac{dx^\sigma}{d\rho}\,,$$

and its chr.inv.-projections are

$$b_\alpha \frac{\mathrm{D}A^\alpha}{d\rho} = \frac{d\varphi}{d\rho} + \frac{1}{c}\left(-F_i q^i \frac{d\tau}{d\rho} + D_{ik} q^i \frac{dx^k}{d\rho}\right),$$

$$h^i_\sigma \frac{\mathrm{D}A^\sigma}{d\rho} = \frac{dq^i}{d\rho} + \left(\frac{\varphi}{c}\frac{dx^k}{d\rho} + q^k \frac{d\tau}{d\rho}\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) - \\ - \frac{\varphi}{c}\, F^i \frac{d\tau}{d\rho} + \Delta^i_{mk}\, q^m \frac{dx^k}{d\rho}\,.$$

The equations of motion of a particle are based on the absolute directional derivative of the particle's world vector. For this reason, the above chr.inv.-projections are the "generic" chr.inv.-equations of motion.

The *divergence* of a tensor field is its "change" along a coordinate axis. Respectively, the *absolute divergence* of an n-dimensional tensor field is its divergence in an n-dimensional space. The divergence of a tensor field is the result of contraction of the field tensor with the absolute derivation operator ∇. The divergence of a vector field A^α is the scalar

$$\nabla_\sigma A^\sigma = \frac{\partial A^\sigma}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu} A^\mu,$$

and the divergence of a field of a 2nd rank tensor, say the tensor $F^{\alpha\beta}$, is the vector

$$\nabla_\sigma F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu} F^{\alpha\mu} + \Gamma^\alpha_{\sigma\mu} F^{\sigma\mu},$$

where, as it can be proved, $\Gamma^\sigma_{\sigma\mu}$ is

$$\Gamma^\sigma_{\sigma\mu} = \frac{\partial \ln\sqrt{-g}}{\partial x^\mu}\,.$$

To prove this, we use the definition of the regular Christoffel symbols (see page 9), which, when re-written with the above indices has the form

$$\Gamma^{\sigma}_{\sigma\mu} = g^{\sigma\rho}\,\Gamma_{\mu\sigma,\rho} = \frac{1}{2}\,g^{\sigma\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\rho}}\right),$$

where, since σ and ρ are free indices here, they can change their sites. As a result, after contracting with the tensor $g^{\sigma\rho}$ the first and the last terms of the above formula for $\Gamma^{\sigma}_{\sigma\mu}$ cancel each other, so the formula for $\Gamma^{\sigma}_{\sigma\mu}$ simplifies as

$$\Gamma^{\sigma}_{\sigma\mu} = \frac{1}{2}\,g^{\sigma\rho}\,\frac{\partial g_{\sigma\rho}}{\partial x^{\mu}}\,.$$

The quantities $g^{\sigma\rho}$ are the components of a tensor reciprocal to the tensor $g_{\sigma\rho}$. For this reason, each component of the matrix $g^{\sigma\rho}$ is formulated as

$$g^{\sigma\rho} = \frac{a^{\sigma\rho}}{g}\,, \qquad g = \det\left\|g_{\sigma\rho}\right\|,$$

where $a^{\sigma\rho}$ is the algebraic co-factor of the matrix element with indices $\sigma\rho$, equal to $(-1)^{\sigma+\rho}$, multiplied by the determinant of the matrix obtained by crossing the row and the column with numbers σ and ρ out of the matrix $g_{\sigma\rho}$. As a result, we obtain $a^{\sigma\rho} = g\,g^{\sigma\rho}$.

Because the determinant of the fundamental metric tensor by definition is formulated as

$$g = \det\left\|g_{\sigma\rho}\right\| = \sum_{\alpha_0\ldots\alpha_3}(-1)^{N(\alpha_0\ldots\alpha_3)}\,g_{0(\alpha_0)}\,g_{1(\alpha_1)}\,g_{2(\alpha_2)}\,g_{3(\alpha_3)}\,,$$

then the quantity dg is $dg = a^{\sigma\rho}dg_{\sigma\rho} = g\,g^{\sigma\rho}dg_{\sigma\rho}$, or

$$\frac{dg}{g} = g^{\sigma\rho}dg_{\sigma\rho}\,.$$

Integrating the left hand side gives $\ln(-g)$, because the g is negative while logarithm is determined for only positive functions. Then, we have $d\ln(-g) = \frac{dg}{g}$. Taking into account that $\sqrt{-g} = \frac{1}{2}\ln(-g)$, we obtain

$$d\ln\sqrt{-g} = \frac{1}{2}\,g^{\sigma\rho}dg_{\sigma\rho}\,,$$

so the above $\Gamma^{\sigma}_{\sigma\mu}$ takes the form

$$\Gamma^{\sigma}_{\sigma\mu} = \frac{1}{2}\,g^{\sigma\rho}\,\frac{\partial g_{\sigma\rho}}{\partial x^{\mu}} = \frac{\partial\ln\sqrt{-g}}{\partial x^{\mu}}\,,$$

which was to be proved.

The divergence of a vector field A^{α} is a scalar quantity. Hence $\nabla_{\sigma}A^{\sigma}$ cannot be projected onto a time line and a spatial section. But this is enough to express $\nabla_{\sigma}A^{\sigma}$ with the chr.inv.-projections of the A^{α} and the physically observable properties of the observer's reference space. Besides that, the regular derivation operators must be replaced with the chr.inv.-derivation operators.

Assuming the above notation φ and q^i for the chr.inv.-projections of the vector A^{α}, we express the remaining components of the A^{α} with them. Then, substituting the regular derivation operators expressed with the chr.inv.-derivation operators (marked by asterisk, see their definition on page 7)

$$\frac{\partial}{\partial t} = \sqrt{g_{00}}\,\frac{^*\partial}{\partial t}\,, \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,,$$

$$\frac{\partial}{\partial x^i} = \frac{^*\partial}{\partial x^i} - \frac{1}{c^2}\,v_i\,\frac{^*\partial}{\partial t}\,,$$

into the general formula for $\nabla_{\sigma}A^{\sigma}$ (page 17) and taking into account that $\sqrt{-g} = \sqrt{h}\,\sqrt{g_{00}}$, after some algebra we obtain the $\nabla_{\sigma}A^{\sigma}$ in the extended chr.inv.-form

$$\nabla_{\sigma}A^{\sigma} = \frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t} + \varphi D\right) + \frac{^*\partial q^i}{\partial x^i} + q^i\,\frac{^*\partial\ln\sqrt{h}}{\partial x^i} - \frac{1}{c^2}\,F_i\,q^i.$$

In the third term of this formula, the quantity

$$\frac{^*\partial\ln\sqrt{h}}{\partial x^i} = \Delta^j_{ji}$$

stands for the chr.inv.-Christoffel symbols Δ^k_{ji} contracted by two indices. Therefore, by analogy with the definition of the absolute divergence of a four-dimensional vector field A^{α} (see page 17), Zelmanov called the quantity

$$^*\nabla_i\,q^i = \frac{^*\partial q^i}{\partial x^i} + q^i\,\frac{^*\partial\ln\sqrt{h}}{\partial x^i} = \frac{^*\partial q^i}{\partial x^i} + q^i\Delta^j_{ji}$$

the *chr.inv.-divergence* of a three-dimensional chr.inv.-vector field q^i. Thus the $\nabla_{\sigma}A^{\sigma}$ takes the final chr.inv.-form

$$\nabla_{\sigma}A^{\sigma} = \frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t} + \varphi D\right) + {}^*\nabla_i\,q^i - \frac{1}{c^2}\,F_i\,q^i.$$

The first term of this formula has no equivalent. It is made up of two parts. The first part is the observable change in time of the time projection φ of the vector A^{α}. The second part φD, since the spur (trace) $D = h^{ik}D_{ik}$ of the chr.inv.-tensor D_{ik} is the observable rate of relative expansion or compression of an elementary volume of the observer's space, is the observable change of the elementary volume of the three-dimensional observable vector field q^i in time.

The difference between the last two terms of this formula, which make up the chr.inv.-quantity

$$^*\widetilde{\nabla}_i\,q^i = {}^*\nabla_i\,q^i - \frac{1}{c^2}\,F_i\,q^i,$$

Zelmanov called the *physical chr.inv.-divergence*, because the chr.inv.-quantity $^*\widetilde{\nabla}_i\,q^i$ takes into account the fact that, in a real physical space, the flow of time is different on the opposite walls of an elementary volume.

Generally speaking, when calculating the divergence of a field we consider an elementary volume of the space, so we calculate the difference between the amounts of a "substance" which flows in and out of the volume over an elementary time interval. The gravitational inertial force F^i results in a different flow of time at different points: the beginnings as well as the ends of the time intervals measured on the opposite walls of a volume will not coincide, which makes these time intervals inapplicable for comparison. Synchronization of clocks on the opposite walls of the volume will give the true result: the measured time intervals will be different. That is, the physical chr.inv.-divergence ${}^*\widetilde{\nabla}_i\, q^i$ is a physical observable in the observer's three-dimensional reference space, which is analogous to a regular divergence.

Next we deduce the chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{\sigma\alpha}$ of an antisymmetric tensor $F^{\alpha\beta} = -F^{\beta\alpha}$

$$\nabla_\sigma F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu} F^{\alpha\mu} + \Gamma^\alpha_{\sigma\mu} F^{\sigma\mu} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \frac{\partial \ln\sqrt{-g}}{\partial x^\mu} F^{\alpha\mu},$$

we need to obtain Maxwell's equations in chr.inv.-form. Here in this formula, the third term $\Gamma^\alpha_{\sigma\mu} F^{\sigma\mu}$ is zero, because contracting the Christoffel symbols $\Gamma^\alpha_{\sigma\mu}$ (they are symmetric by their lower indices) with the antisymmetric tensor $F^{\sigma\mu}$ gives zero as in the case of any symmetric and antisymmetric geometric objects.

The quantity $\nabla_\sigma F^{\sigma\alpha}$ is a four-dimensional vector, therefore its chr.inv.-projections are

$$T = b_\alpha \nabla_\sigma F^{\sigma\alpha}, \quad B^i = h^i_\alpha \nabla_\sigma F^{\sigma\alpha} = \nabla_\sigma F^{\sigma i}.$$

Denoting the chr.inv.-projections of the tensor $F^{\alpha\beta}$ as

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}, \quad H^{ik} = F^{ik},$$

we obtain the remaining non-zero components of the $F^{\alpha\beta}$ expressed with its chr.inv.-projections

$$F_{0\cdot}^{\cdot 0} = \frac{1}{c}\, v_k E^k,$$

$$F_{k\cdot}^{\cdot 0} = \frac{1}{\sqrt{g_{00}}}\left(E_i - \frac{1}{c}\, v_n H_{k\cdot}^{\cdot n} - \frac{1}{c^2}\, v_k v_n E^n\right),$$

$$F^{0i} = \frac{E^i - \frac{1}{c}\, v_k H^{ik}}{\sqrt{g_{00}}},$$

$$F_{0i} = -\sqrt{g_{00}}\, E_i\,,$$

$$F_{i\cdot}^{\cdot k} = -H_{i\cdot}^{\cdot k} - \frac{1}{c}\, v_i E^k,$$

$$F_{ik} = H_{ik} + \frac{1}{c}\left(v_i E_k - v_k E_i\right),$$

and also the square of the tensor $F^{\alpha\beta}$ in the form as well expressed with its chr.inv.-projections

$$F_{\alpha\beta} F^{\alpha\beta} = H_{ik} H^{ik} - 2 E_i E^i.$$

Substituting these formulae into the above general formulae for T and B^i and then replacing the regular derivation operators with the chr.inv.-derivation operators, after some algebra we obtain the formulae for the chr.inv.-projections T and B^i of the absolute divergence $\nabla_\sigma F^{\sigma\alpha}$ of the antisymmetric tensor $F^{\alpha\beta} = -F^{\beta\alpha}$ in detail

$$T = \frac{\nabla_\sigma F_{0\cdot}^{\cdot\sigma}}{\sqrt{g_{00}}} = \frac{{}^*\partial E^i}{\partial x^i} + E^i\, \frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} - \frac{1}{c}\, H^{ik} A_{ik}\,,$$

$$B^i = \nabla_\sigma F^{\sigma i} = \frac{{}^*\partial H^{ik}}{\partial x^k} + H^{ik}\frac{{}^*\partial \ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2}\, F_k H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + D E^i\right).$$

Taking into account that

$$\frac{{}^*\partial E^i}{\partial x^i} + E^i\, \frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} = {}^*\nabla_i\, E^i$$

is the chr.inv.-divergence of the vector E^i, and also that

$$\frac{{}^*\partial H^{ik}}{\partial x^k} + H^{ik}\frac{{}^*\partial \ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2}\, F_k H^{ik} = {}^*\nabla_k H^{ik} - \frac{1}{c^2}\, F_k H^{ik} = {}^*\widetilde{\nabla}_k\, H^{ik}$$

is the physical chr.inv.-divergence of the tensor H^{ik}, we arrive at the final formulae for chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{\sigma\alpha}$ of the antisymmetric tensor $F^{\alpha\beta}$

$$T = {}^*\nabla_i\, E^i - \frac{1}{c}\, H^{ik} A_{ik}\,,$$

$$B^i = {}^*\widetilde{\nabla}_k\, H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + D E^i\right).$$

Calculate the chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{*\sigma\alpha}$ of the pseudotensor $F^{*\alpha\beta}$ dual to the antisymmetric tensor $F^{\alpha\beta}$. For such a dual pseudotensor we have

$$F^{*\alpha\beta} = \frac{1}{2}\, E^{\alpha\beta\mu\nu} F_{\mu\nu}\,, \quad F_{*\alpha\beta} = \frac{1}{2}\, E_{\alpha\beta\mu\nu} F^{\mu\nu}.$$

Denoting its chr.inv.-projections as

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}}\,, \qquad E^{*ik} = F^{*ik},$$

we see that the obvious relations $H^{*i} \sim H^{ik}$ and $E^{*ik} \sim E^i$ exist between the chr.inv.-projections of the antisymmetric tensor $F^{\alpha\beta}$ and the pseudotensor $F^{*\alpha\beta}$, which are due to the duality of these tensors to each other.

As a result of these relations, given that

$$\frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{1}{2}\, \varepsilon^{ipq} H_{pq}\,, \quad F^{*ik} = -\,\varepsilon^{ikp} E_p\,,$$

the remaining components of the pseudotensor $F^{*\alpha\beta}$, formulated with the chr.inv.-projections of its dual tensor $F^{\alpha\beta}$ have the following form

$$F_{0\cdot}^{*\cdot 0} = \frac{1}{2c}\, v_k\, \varepsilon^{kpq}\left[H_{pq} + \frac{1}{c}\left(v_p E_q - v_q E_p\right)\right],$$

$$F_{i\cdot}^{*\cdot 0} = \frac{1}{2\sqrt{g_{00}}}\left[\varepsilon_{i\cdot}^{\cdot pq} H_{pq} + \frac{1}{c}\,\varepsilon_{i\cdot}^{\cdot pq}\left(v_p E_q - v_q E_p\right) - \right.$$
$$\left. - \frac{1}{c^2}\,\varepsilon^{kpq} v_i v_k H_{pq} - \frac{1}{c^3}\,\varepsilon^{kpq} v_i v_k \left(v_p E_q - v_q E_p\right)\right],$$

$$F^{*0i} = \frac{1}{2\sqrt{g_{00}}}\,\varepsilon^{ipq}\left[H_{pq} + \frac{1}{c}\left(v_p E_q - v_q E_p\right)\right],$$

$$F_{*0i} = \frac{1}{2}\sqrt{g_{00}}\;\varepsilon_{ipq} H^{pq},$$

$$F_{i\cdot}^{*\cdot k} = \varepsilon_{i\cdot}^{\cdot kp} E_p - \frac{1}{2c}\, v_i\, \varepsilon^{kpq} H_{pq} - \frac{1}{c^2}\, v_i v_m\, \varepsilon^{mkp} E_p\,,$$

$$F_{*ik} = \varepsilon_{ikp}\left(E^p - \frac{1}{c}\, v_q H^{pq}\right),$$

while the square of the pseudotensor $F^{*\alpha\beta}$ has the form

$$F_{*\alpha\beta} F^{*\alpha\beta} = \varepsilon^{ipq}\left(E_p H_{iq} - E_i H_{pq}\right).$$

With the above components, after some algebra we obtain the chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{*\sigma\alpha}$ of the dual pseudotensor $F^{*\alpha\beta}$ in detail

$$\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} = \frac{{}^*\partial H^{*i}}{\partial x^i} + H^{*i}\,\frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} - \frac{1}{c}\, E^{*ik} A_{ik}\,,$$

$$\nabla_\sigma F^{*\sigma i} = \frac{{}^*\partial E^{*ik}}{\partial x^i} + E^{*ik}\,\frac{{}^*\partial \ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2}\, F_k E^{*ik} - $$
$$- \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + D H^{*i}\right),$$

then, using the formulae for the chr.inv.-divergence ${}^*\nabla_i H^{*i}$ and the physical chr.inv.-divergence ${}^*\widetilde{\nabla}_k E^{*ik}$, we arrive at the final formulae for chr.inv.-projections of the absolute divergence $\nabla_\sigma F^{*\sigma\alpha}$ of the dual pseudotensor $F^{*\alpha\beta}$

$$\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} = {}^*\nabla_i H^{*i} - \frac{1}{c}\, E^{*ik} A_{ik}\,,$$

$$\nabla_\sigma F^{*\sigma i} = {}^*\widetilde{\nabla}_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + D H^{*i}\right).$$

Apart from the absolute divergence of vectors, antisymmetric tensors and pseudotensors of the 2nd rank, we need to deduce the chr.inv.-projections of the absolute divergence of a symmetric tensor of the 2nd rank (we need them to obtain the conservation law in chr.inv.-form).

Just as Zelmanov did, we denote the chr.inv.-projections of a symmetric tensor $T^{\alpha\beta}$ as

$$\frac{T_{00}}{g_{00}} = \rho\,, \qquad \frac{T_0^i}{\sqrt{g_{00}}} = K^i, \qquad T^{ik} = N^{ik},$$

whence, following the same algebra as above, we obtain the chr.inv.-projections of the absolute divergence $\nabla_\sigma T^{\sigma\alpha}$ of the symmetric tensor $T^{\alpha\beta}$ in detail

$$\frac{\nabla_\sigma T_0^\sigma}{\sqrt{g_{00}}} = \frac{{}^*\partial\rho}{\partial t} + \rho D + D_{ik} N^{ik} + c\,{}^*\nabla_i K^i - \frac{2}{c}\, F_i K^i,$$

$$\nabla_\sigma T^{\sigma i} = c\,\frac{{}^*\partial K^i}{\partial t} + c D K^i + 2c\left(D_k^i + A_{k\cdot}^{\cdot i}\right) K^k + $$
$$+ c^2\,{}^*\nabla_k N^{ik} - F_k N^{ik} - \rho F^i.$$

In addition to the inner (scalar) product of a tensor with the absolute differentiation operator ∇, which is the absolute divergence of this tensor field, there may also be a difference between the covariant derivatives of the field. This quantity is known as the *curl* of the field, because from a geometric point of view it is the vortex (rotation) of the field itself. The *absolute curl* is the curl of an n-dimensional tensor field in an n-dimensional space.

The curl of an arbitrary four-dimensional vector field A^α is a covariant antisymmetric tensor of the 2nd rank*

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\,,$$

where $\nabla_\mu A_\nu$ is the absolute derivative of the A_α with respect to the coordinate x^μ

$$\nabla_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} - \Gamma_{\nu\mu}^\sigma A_\sigma\,.$$

The curl contracted with the four-dimensional absolutely antisymmetric discriminant tensor $E^{\alpha\beta\mu\nu}$ is the pseudotensor

$$F^{*\alpha\beta} = E^{\alpha\beta\mu\nu}\left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right) = E^{\alpha\beta\mu\nu}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right).$$

In electrodynamics, the electromagnetic field tensor $F_{\mu\nu}$ (Maxwell's tensor) is the curl of the four-dimensional electromagnetic field potential A^α. Therefore, we need the formulae for the chr.inv.-projections of the four-dimensional curl $F_{\mu\nu}$ and its dual pseudotensor $F^{*\alpha\beta}$ expressed in terms of the chr.inv.-projections of the four-dimensional vector potential A^α that forms them.

After the same algebra as above, we obtain the chr.inv.-projections of the absolute curl $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ expressed in terms of the chr.inv.-projections φ and q^i of the vector A^α forming this curl

$$\frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = \frac{g^{i\alpha} F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik}\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\,\frac{{}^*\partial q_k}{\partial t}\right) - \frac{\varphi}{c^2}\, F^i,$$

$$F^{ik} = g^{i\alpha} g^{k\beta} F_{\alpha\beta} = h^{im} h^{kn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c}\, A^{ik}.$$

*Strictly speaking, a real geometric curl is not a tensor, but its dual pseudotensor. This is because the invariance with respect to reflection is necessary for any rotation. See §98 in the very good textbook *Riemannsche Geometrie und Tensoranalysis* [17] written by Peter Raschewski (1907–1983), the well-known expert in Riemannian geometry.

The remaining components of the curl $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ with taking into account that $F_{00} = F^{00} = 0$ just like for any antisymmetric tensor have the form

$$F_{0i} = \left(1 - \frac{\mathrm{w}}{c^2}\right)\left(\frac{\varphi}{c^2}\,F_i - \frac{{}^*\partial\varphi}{\partial x^i} - \frac{1}{c}\frac{{}^*\partial q_i}{\partial t}\right),$$

$$F_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} + \frac{\varphi}{c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right) + \frac{1}{c}\left(v_i\frac{{}^*\partial\varphi}{\partial x^k} - v_k\frac{{}^*\partial\varphi}{\partial x^i}\right) + \frac{1}{c^2}\left(v_i\frac{{}^*\partial q_k}{\partial t} - v_k\frac{{}^*\partial q_i}{\partial t}\right),$$

$$F_{0\cdot}^{\cdot 0} = -\frac{\varphi}{c^3}\,v_k F^k + \frac{1}{c}\,v^k\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t}\right),$$

$$F_{k\cdot}^{\cdot 0} = -\frac{1}{\sqrt{g_{00}}}\left[\frac{\varphi}{c^2}\,F_k - \frac{{}^*\partial\varphi}{\partial x^k} - \frac{1}{c}\frac{{}^*\partial q_k}{\partial t} + \frac{2\varphi}{c^2}\,v^m A_{mk} + \frac{1}{c^2}\,v_k v^m\left(\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right) - \frac{1}{c}\,v^m\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{\varphi}{c^4}\,v_k v_m F^m\right],$$

$$F_{k\cdot}^{\cdot i} = h^{im}\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{1}{c}\,h^{im}v_k\frac{{}^*\partial\varphi}{\partial x^m} - \frac{1}{c^2}\,h^{im}v_k\frac{{}^*\partial q_m}{\partial t} + \frac{\varphi}{c^3}\,v_k F^i + \frac{2\varphi}{c}\,A_{k\cdot}^{\cdot i},$$

$$F^{0k} = \frac{1}{\sqrt{g_{00}}}\left[h^{km}\left(\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right) - \frac{\varphi}{c^2}\,F^k + \frac{1}{c}\,v^n h^{mk}\left(\frac{{}^*\partial q_n}{\partial x^m} - \frac{{}^*\partial q_m}{\partial x^n}\right) - \frac{2\varphi}{c^2}\,v_m A^{mk}\right].$$

Respectively, the chr.inv.-projections of the dual pseudotensor $F^{*\alpha\beta}$ of the curl $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ have the form

$$\frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{g_{0\alpha}F^{*\alpha i}}{\sqrt{g_{00}}} = \varepsilon^{ikm}\left[\frac{1}{2}\left(\frac{{}^*\partial q_k}{\partial x^m} - \frac{{}^*\partial q_m}{\partial x^k}\right) - \frac{\varphi}{c}\,A_{km}\right],$$

$$F^{*ik} = \varepsilon^{ikm}\left(\frac{\varphi}{c^2}\,F_m - \frac{{}^*\partial\varphi}{\partial x^m} - \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right),$$

where $F_{0\cdot}^{*\cdot i} = g_{0\alpha}F^{*\alpha i} = g_{0\alpha}E^{\alpha i\mu\nu}F_{\mu\nu}$ is calculated using the above components of the curl $F_{\mu\nu}$.

Laplace's operator known also as *Laplacian* is the three-dimensional derivation operator

$$\Delta = \nabla\nabla = \nabla^2 = -g^{ik}\,\nabla_i\nabla_k\,.$$

The four-dimensional generalization of Laplace's operator in a pseudo-Riemannian space is *d'Alembert's operator* known also as *d'Alembertian*

$$\Box = g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta\,.$$

Let us apply d'Alembert's operator to a scalar field and a vector field in the four-dimensional pseudo-Riemannian space (the space-time of General Relativity), and then express the calculation results in chr.inv.-form.

First we apply d'Alembert's operator to a scalar field φ

$$\Box\,\varphi = g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta\,\varphi = g^{\alpha\beta}\frac{\partial\varphi}{\partial x^\alpha}\left(\frac{\partial\varphi}{\partial x^\beta}\right) = g^{\alpha\beta}\frac{\partial^2\varphi}{\partial x^\alpha\partial x^\beta}\,,$$

because in this case the calculation is much simpler: the absolute derivative of a scalar, $\nabla_\alpha\varphi$, does not contain the Christoffel symbols, so it becomes the regular derivative.

We express the components of the fundamental metric tensor in terms of chronometric invariants. For g^{ik} we have $g^{ik} = -h^{ik}$ (see page 5). The components g^{0i} are obtained from the formula for the linear velocity of rotation of the observer's space $v^i = -c\,g^{0i}\sqrt{g_{00}}$ (see page 7)

$$g^{0i} = -\frac{1}{c\sqrt{g_{00}}}\,v^i.$$

The component g^{00} is obtained from the main property of the fundamental metric tensor $g_{\alpha\sigma}g^{\beta\sigma} = g_\alpha^\beta$. Setting up $\alpha = \beta = 0$ in the mentioned property, we obtain

$$g_{0\sigma}g^{0\sigma} = g_{00}\,g^{00} + g_{0i}\,g^{0i} = \delta_0^0 = 1,$$

whence, taking into account that

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2, \quad g_{0i} = -\frac{1}{c}\,v_i\left(1 - \frac{\mathrm{w}}{c^2}\right),$$

we obtain the formula

$$g^{00} = \frac{1}{\left(1 - \frac{\mathrm{w}}{c^2}\right)^2}\left(1 - \frac{1}{c^2}\,v_i v^i\right), \quad v_i v^i = h_{ik}v^i v^k = v^2.$$

Substituting the obtained formulae for g^{00}, g^{0i} and g^{ik} into the above general formula for $\Box\,\varphi$ and then replacing the regular derivation operators with the chr.inv.-derivation operators, we obtain the d'Alembertian of the scalar field φ in chr.inv.-form

$$\Box\,\varphi = \frac{1}{c^2}\frac{{}^*\partial^2\varphi}{\partial t^2} - h^{ik}\frac{{}^*\partial^2\varphi}{\partial x^i\partial x^k} = {}^*\Box\,\varphi,$$

where ${}^*\Box$ is the chr.inv.-d'Alembert operator, and ${}^*\Delta$ is the chr.inv.-Laplace operator

$${}^*\Box = \frac{1}{c^2}\frac{{}^*\partial^2}{\partial t^2} - h^{ik}\frac{{}^*\partial^2}{\partial x^i\partial x^k} = \frac{1}{c^2}\frac{{}^*\partial^2}{\partial t^2} - {}^*\Delta,$$

$${}^*\Delta = h^{ik}\frac{{}^*\partial^2}{\partial x^i\partial x^k} = -g^{ik}\,{}^*\nabla_i\,{}^*\nabla_k\,.$$

Now, we apply d'Alembert's operator to an arbitrary four-dimensional vector field A^α

$$\Box A^\alpha = g^{\mu\nu}\,\nabla_\mu\nabla_\nu A^\alpha.$$

Because $\Box A^{\alpha}$ is a four-dimensional vector, the chr.inv.-projections of it are

$$T = b_{\sigma}\Box A^{\sigma} = b_{\sigma}\, g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}A^{\sigma},$$

$$B^{i} = h^{i}_{\sigma}\Box A^{\sigma} = h^{i}_{\sigma}\, g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}A^{\sigma}.$$

It should be noted that the derivation of the d'Alembertian of a vector field in a Riemannian space is not a trivial task. This is because in this case, the Christoffel symbols are not zeroes and, therefore, the formulae for the chr.inv.-projections of the second derivatives take many pages*.

So, after some difficult algebra we had obtained formulae for the chr.inv.-projections of the d'Alembertian of the vector field A^{α} in the four-dimensional pseudo-Riemannian space. They have the following form†

$$\begin{aligned} T = {}^{*}\Box\varphi - \frac{1}{c^{3}}\frac{{}^{*}\partial}{\partial t}\left(F_{k}q^{k}\right) - \frac{1}{c^{3}}F_{i}\frac{{}^{*}\partial q^{i}}{\partial t} + \frac{1}{c^{2}}F^{i}\frac{{}^{*}\partial\varphi}{\partial x^{i}} + \\ + h^{ik}\Delta^{m}_{ik}\frac{{}^{*}\partial\varphi}{\partial x^{m}} - h^{ik}\frac{1}{c}\frac{{}^{*}\partial}{\partial x^{i}}\Big[\left(D_{kn}+A_{kn}\right)q^{n}\Big] + \frac{D}{c^{2}}\frac{{}^{*}\partial\varphi}{\partial t} - \\ - \frac{1}{c}D^{k}_{m}\frac{{}^{*}\partial q^{m}}{\partial x^{k}} + \frac{2}{c^{3}}A_{ik}F^{i}q^{k} + \frac{\varphi}{c^{4}}F_{i}F^{i} - \frac{\varphi}{c^{2}}D_{mk}D^{mk} - \\ - \frac{D}{c^{3}}F_{m}q^{m} - \frac{1}{c}\Delta^{m}_{kn}D^{k}_{m}q^{n} + \frac{1}{c}h^{ik}\Delta^{m}_{ik}\left(D_{mn}+A_{mn}\right)q^{n}, \end{aligned}$$

$$\begin{aligned} B^{i} = {}^{*}\Box A^{i} + \frac{1}{c^{2}}\frac{{}^{*}\partial}{\partial t}\Big[\big(D^{i}_{k}+A_{k\cdot}^{\cdot i}\big)q^{k}\Big] + \frac{D}{c^{2}}\frac{{}^{*}\partial q^{i}}{\partial t} + \\ + \frac{1}{c^{2}}\big(D^{i}_{k}+A_{k\cdot}^{\cdot i}\big)\frac{{}^{*}\partial q^{k}}{\partial t} - \frac{1}{c^{3}}\frac{{}^{*}\partial}{\partial t}\left(\varphi F^{i}\right) - \frac{1}{c^{3}}F^{i}\frac{{}^{*}\partial\varphi}{\partial t} + \\ + \frac{1}{c^{2}}F^{k}\frac{{}^{*}\partial q^{i}}{\partial x^{k}} - \frac{1}{c}\left(D^{mi}+A^{mi}\right)\frac{{}^{*}\partial\varphi}{\partial x^{m}} + \frac{1}{c^{4}}q^{k}F_{k}F^{i} + \\ + \frac{1}{c^{2}}\Delta^{i}_{km}q^{m}F^{k} - \frac{\varphi}{c^{3}}DF^{i} + \frac{D}{c^{2}}\big(D^{i}_{n}+A_{n\cdot}^{\cdot i}\big)q^{n} - \\ - h^{km}\left\{\frac{{}^{*}\partial}{\partial x^{k}}\left(\Delta^{i}_{mn}q^{n}\right) + \frac{1}{c}\frac{{}^{*}\partial}{\partial x^{k}}\Big[\varphi\big(D^{i}_{m}+A_{m\cdot}^{\cdot i}\big)\Big] + \right. \\ + \left(\Delta^{i}_{kn}\Delta^{n}_{mp} - \Delta^{n}_{km}\Delta^{i}_{np}\right)q^{p} + \frac{\varphi}{c}\Big[\Delta^{i}_{kn}\big(D^{n}_{m}+A_{m\cdot}^{\cdot n}\big) - \\ \left. - \Delta^{n}_{km}\big(D^{i}_{n}+A_{n\cdot}^{\cdot i}\big)\Big] + \Delta^{i}_{kn}\frac{{}^{*}\partial q^{n}}{\partial x^{m}} - \Delta^{n}_{km}\frac{{}^{*}\partial q^{i}}{\partial x^{n}}\right\}, \end{aligned}$$

where ${}^{*}\Box\varphi$ and ${}^{*}\Box q^{i}$ are the result of applying the chr.inv.-d'Alembert operator to the quantities $\varphi = \frac{A_0}{\sqrt{g_{00}}}$ and $q^{i} = A^{i}$, which are chr.inv.-projections (physically observable components) of the vector A^{α}

$${}^{*}\Box\varphi = \frac{1}{c^{2}}\frac{{}^{*}\partial^{2}\varphi}{\partial t^{2}} - h^{ik}\frac{{}^{*}\partial^{2}\varphi}{\partial x^{i}\partial x^{k}},$$

$${}^{*}\Box q^{i} = \frac{1}{c^{2}}\frac{{}^{*}\partial^{2}q^{i}}{\partial t^{2}} - h^{km}\frac{{}^{*}\partial^{2}q^{i}}{\partial x^{k}\partial x^{m}}.$$

The main criterion for correct calculations in such a complicated case as here is Zelmanov's rule of the chronometric invariance: "Correct calculations make all terms in the final equations chronometrically invariant quantities. That is to say, the final equations consist of the chr.inv.-quantities, their chr.inv.-derivatives, and also the chr.inv.-properties of the observer's reference space. If at least one error was made in the calculations, then some terms of the final equations will not be chronometric invariants."

In the Galilean reference frame in the Minkowski space (the space-time of Special Relativity), Laplace's and d'Alembert's operators take the simplified form

$$\Delta = \frac{\partial^{2}}{\partial x^{1}\partial x^{1}} + \frac{\partial^{2}}{\partial x^{2}\partial x^{2}} + \frac{\partial^{2}}{\partial x^{3}\partial x^{3}},$$

$$\Box = \frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}} - \frac{\partial^{2}}{\partial x^{1}\partial x^{1}} - \frac{\partial^{2}}{\partial x^{2}\partial x^{2}} - \frac{\partial^{2}}{\partial x^{3}\partial x^{3}} = \frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}} - \Delta.$$

D'Alembert's operator applied to a tensor field and equated to zero or not zero, gives the *d'Alembert equations* for this field. From a physical point of view, these are the equations of propagation of waves of the field. If the d'Alembertian of a field is not zero, these are the equations of propagation of the waves enforced by the sources that induce this field; they are called the *d'Alembert equations with sources*. For instance, the sources of electromagnetic fields are electric charges and currents. If the d'Alembertian of a field is zero, then these are the equations of propagation of waves in the field not related to any sources. If the space-time region under consideration, in addition to the tensor field, is filled with another medium, then the d'Alembert equations gain an additional term characterizing this medium (this term can be found using the equations which determine the medium).

These are the basics of tensor calculus expressed in terms of chronometric invariants.

Next we present formulae for the most common equations used in General Relativity, in the form expressed in terms of physical observables (chronometric invariants).

First, consider the equations of motion of a particle. A particle under the influence of gravitation only falls freely and thus travels along the shortest (*geodesic*) line. Such motion is called *free* or *geodesic motion*. If an additional non-gravitational force also acts on the particle, then the force deviates this particle from its geodesic trajectory, and the motion becomes *non-geodesic*.

From a geometric point of view, motion of a particle in the four-dimensional pseuso-Riemannian space (space-time)

*This is one of the reasons why applications of the theory of electromagnetic fields are calculated in the Galilean reference frame in the Minkowski space (the space-time of Special Relativity), where the Christoffel symbols are zeroes. General covariant notation hardly allows unambiguous interpretation of calculation results, unless they are formulated with physical observable quantities (chronometric invariants) or demoted to a simple specific case like that in the Minkowski space, for instance.

†The above chr.inv.-projections of the d'Alembertian of a vector field in the four-dimensional pseudo-Riemannian space were deduced not by Zelmanov, but by one of us, L. Borisova, in the 1980s.

is parallel transport of the four-dimensional vector Q^α, which is tangential to the particle's trajectory at any of its points and completely characterizes this particle. Therefore, the equations of motion of a particle actually determine the parallel transport of the particle's vector Q^α along the particle's four-dimensional trajectory and they are the equations of the absolute derivative of this vector with respect to a parameter ρ, which is non-zero along the trajectory

$$\frac{\mathrm{D}Q^\alpha}{d\rho} = \frac{dQ^\alpha}{d\rho} + \Gamma^\alpha_{\mu\nu} Q^\mu \frac{dx^\nu}{d\rho},$$

where $\mathrm{D}Q^\alpha = dQ^\alpha + \Gamma^\alpha_{\mu\nu} Q^\mu dx^\nu$ is the absolute differential of the transported vector Q^α (i.e., its absolute increment) along the trajectory.

If a particle travels along a geodesic trajectory (free motion), then the particle's characteristic vector is transported in Levi-Civita's sense: the square of the transported vector remains unchanged $Q_\alpha Q^\alpha = const$ along the trajectory, while the absolute derivative of the transported vector is zero and such equations are called the *equations of free motion*.

A mass-bearing particle (such particles travel along non-isotropic space-time trajectories) is characterized by its own four-dimensional momentum vector

$$P^\alpha = m_0 \frac{dx^\alpha}{ds}, \quad P_\alpha P^\alpha = m_0^2 = const,$$

where m_0 is the particle's rest-mass. Respectively, the equations of motion of a free mass-bearing particle are

$$\frac{dP^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} P^\mu \frac{dx^\nu}{ds} = 0.$$

A massless light-like particle (such particles travel along isotropic space-time trajectories) is characterized by its own four-dimensional wave vector

$$K^\alpha = \frac{\omega}{c}\frac{dx^\alpha}{d\sigma}, \quad K_\alpha K^\alpha = 0,$$

where ω is the characteristic frequency of the massless particle, and $d\sigma = h_{ik}\, dx^i dx^k$ is the three-dimensional chr.inv.-interval, which, since $ds^2 = c^2 d\tau^2 - d\sigma^2 = 0$ along isotropic trajectories, is invariant along them. Respectively, the equations of motion of a free massless (light-like) particle are

$$\frac{dK^\alpha}{d\sigma} + \Gamma^\alpha_{\mu\nu} K^\mu \frac{dx^\nu}{d\sigma} = 0.$$

The projections of the above four-dimensional equations of motion onto the time line and the three-dimensional spatial section of an observer are, respectively, the chr.inv.-equations of motion of a free mass-bearing particle

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i + \frac{m}{c^2} D_{ik} \mathrm{v}^i \mathrm{v}^k = 0,$$

$$\frac{d(m\mathrm{v}^i)}{d\tau} + 2m(D^i_k + A^{\cdot i}_{k\cdot})\mathrm{v}^k - mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0,$$

and the chr.inv.-equations of motion of a free massless (light-like) particle

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_i c^i + \frac{\omega}{c^2} D_{ik} c^i c^k = 0,$$

$$\frac{d(\omega c^i)}{d\tau} + 2\omega(D^i_k + A^{\cdot i}_{k\cdot})c^k - \omega F^i + \omega\Delta^i_{nk} c^n c^k = 0,$$

where m is the relativistic mass of the travelling mass-bearing particle, ω is the characteristic frequency of the massless particle, $d\tau$ is the physically observable time interval, and v^i is the chr.inv.-vector of the physically observable velocity of the mass-bearing particle. Along isotropic trajectories (trajectories of light) the v^i transforms into the chr.inv.-vector of the physically observable velocity of light, the square of which is $c_i c^i = h_{ik} c^i c^k = c^2$ (see page 6).

If a particle travels along a non-geodesic trajectory, then $Q_\alpha Q^\alpha \neq const$, and the absolute derivative of the transported vector Q^α is equal to a force Φ^α that deviates the particle from a geodesic line. Such equations are called the *equations of non-geodesic motion* [5]. In this case, the right hand side of the above chr.inv.-equations of motion is different from zero and contains the respective chr.inv.-projections of the deviating force Φ^α.

The chr.inv.-equations of motion show how the observed motion of particles depends on the physically observable gravitational inertial force F^i, rotation A_{ik}, deformation D_{ik} and inhomogeneity (the coherence coefficients Δ^i_{kn}) of the observer's reference space.

Let us now turn to the basics of electrodynamics in the four-dimensional pseudo-Riemannian space.

The electromagnetic field tensor $F^{\mu\nu}$ is determined as the curl $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ of the four-dimensional electromagnetic field potential A^α. Following the terminology of electrodynamics, we call the chr.inv.-projections of the A^α (page 17) the chr.inv.-scalar potential φ and the chr.inv.-vector potential q^i of the electromagnetic field

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}, \qquad q^i = A^i,$$

and the chr.inv.-projections of the electromagnetic field tensor $F^{\mu\nu}$ (page 20) — the chr.inv.-electric strength E^i and the chr.inv.-magnetic strength H^{ik} of the field

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = \frac{g^{i\alpha}F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik}\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t}\right) - \frac{\varphi}{c^2}F^i,$$

$$H^{ik} = F^{ik} = g^{i\alpha}g^{k\beta}F_{\alpha\beta} = h^{im}h^{kn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c}A^{ik},$$

where their covariant (lower-index) versions are

$$E_i = h_{ik}E^k = \frac{{}^*\partial\varphi}{\partial x^i} + \frac{1}{c}\frac{{}^*\partial q_i}{\partial t} - \frac{\varphi}{c^2}F_i,$$

$$H_{ik} = h_{im}h_{kn}H^{mn} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} - \frac{2\varphi}{c}A_{ik},$$

and the mixed components $H_{k\cdot}^{\cdot m} = -H_{\cdot k}^{m\cdot}$ are obtained from H^{ik} using the metric chr.inv.-tensor h_{ik}, i.e., $H_{k\cdot}^{\cdot m} = h_{ki}H^{im}$.

Respectively, the electromagnetic field pseudotensor $F^{*\alpha\beta}$ dual to the field tensor, i.e., $F^{*\alpha\beta} = \frac{1}{2}E^{\alpha\beta\mu\nu}F_{\mu\nu}$, has the following chr.inv.-projections

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{1}{2}\varepsilon^{imn}\left(\frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m} - \frac{2\varphi}{c}A_{mn}\right) = \frac{1}{2}\varepsilon^{imn}H_{mn}\,,$$

$$E^{*ik} = F^{*ik} = \varepsilon^{ikn}\left(\frac{\varphi}{c^2}F_n - \frac{^*\partial\varphi}{\partial x^n} - \frac{1}{c}\frac{^*\partial q_n}{\partial t}\right) = -\varepsilon^{ikn}E_n\,,$$

which we call the chr.inv.-magnetic strength pseudovector H^{*i} and the chr.inv.-electric strength pseudotensor E^{*ik}. It is obvious that the quantities H^{*i} and H_{mn} are dually conjugate, and the quantities E^{*ik} and E_m are also dually conjugate.

The above formulae show that the observed electric and magnetic strengths of the electromagnetic field depend on the physically observable gravitational inertial force F^i and rotation A_{ik} of the observer's reference space.

So forth, the electromagnetic field invariants

$$J_1 = F_{\mu\nu}F^{\mu\nu} = H_{ik}H^{ik} - 2E_iE^i = -2\left(E_iE^i - H_{*i}H^{*i}\right),$$

$$J_2 = F_{\mu\nu}F^{*\mu\nu} = \varepsilon^{imn}\left(E_mH_{in} - E_iH_{nm}\right) = -4E_iH^{*i},$$

the first of which is a scalar, and the second is a pseudoscalar, have the following detailed chr.inv.-formulation

$$J_1 = 2\left[h^{im}h^{kn}\left(\frac{^*\partial q_i}{\partial x^k} - \frac{^*\partial q_k}{\partial x^i}\right)\frac{^*\partial q_m}{\partial x^n} - h^{ik}\frac{^*\partial\varphi}{\partial x^i}\frac{^*\partial\varphi}{\partial x^k} - \right.$$
$$- \frac{2}{c}h^{ik}\frac{^*\partial\varphi}{\partial x^i}\frac{^*\partial q_k}{\partial t} - \frac{1}{c^2}h^{ik}\frac{^*\partial q_i}{\partial t}\frac{^*\partial q_k}{\partial t} + \frac{8\varphi}{c^2}\Omega_{*i}\Omega^{*i} -$$
$$\left. - \frac{2\varphi}{c}\varepsilon^{imn}\Omega_{*m}\frac{^*\partial q_i}{\partial x^n} + \frac{2\varphi}{c^2}\frac{^*\partial\varphi}{\partial x^i}F^i + \frac{2\varphi}{c^3}\frac{^*\partial q_i}{\partial t}F^i - \frac{\varphi}{c^4}F_iF^i\right],$$

$$J_2 = \frac{1}{2}\left[\varepsilon^{imn}\left(\frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m}\right) - \frac{4\varphi}{c}\Omega^{*i}\right]\times$$
$$\times\left(\frac{^*\partial\varphi}{\partial x^i} + \frac{1}{c}\frac{^*\partial q_i}{\partial t} - \frac{\varphi}{c^2}F_i\right).$$

Mathematically, any electromagnetic field in the four-dimensional pseudo-Riemannian space is completely characterized by a system of 10 equations in 10 unknowns. First, this system includes Maxwell's equations

$$\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c}j^\mu, \qquad \nabla_\sigma F^{*\mu\sigma} = 0\,,$$

the chr.inv.-projections of which give two groups of equations, which we call the chr.inv.-Maxwell equations* and which have the following form

$$\left.\begin{aligned} &{}^*\nabla_i E^i - \frac{1}{c}H^{ik}A_{ik} = 4\pi\rho \\ &{}^*\nabla_k H^{ik} - \frac{1}{c^2}F_kH^{ik} - \frac{1}{c}\left(\frac{^*\partial E^i}{\partial t} + DE^i\right) = \frac{4\pi}{c}j^i \end{aligned}\right\}\ \text{I}\,,$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} - \frac{1}{c}E^{*ik}A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2}F_kE^{*ik} - \frac{1}{c}\left(\frac{^*\partial H^{*i}}{\partial t} + DH^{*i}\right) = 0 \end{aligned}\right\}\ \text{II}\,,$$

or, in another notation

$$\left.\begin{aligned} &{}^*\nabla_i E^i - \frac{2}{c}\Omega_{*m}H^{*m} = 4\pi\rho \\ &\varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\left(H_{*m}\sqrt{h}\right) - \frac{1}{c}\frac{^*\partial}{\partial t}\left(E^i\sqrt{h}\right) = \frac{4\pi}{c}j^i\sqrt{h} \end{aligned}\right\}\ \text{I}\,,$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} + \frac{2}{c}\Omega_{*m}E^m = 0 \\ &\varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\left(E_m\sqrt{h}\right) + \frac{1}{c}\frac{^*\partial}{\partial t}\left(H^{*i}\sqrt{h}\right) = 0 \end{aligned}\right\}\ \text{II}\,.$$

These are 8 equations in 10 unknowns, which are 3 components of the chr.inv.-electric strengths E^i, 3 components of the chr.inv.-magnetic strength H^{*i}, 1 component of the electric charge density ρ and 3 components of the chr.inv.-current density vector j^i. The latter two, known as the electromagnetic field sources, are the chr.inv.-projections

$$\rho = \frac{1}{c}b^\alpha j_\alpha = \frac{1}{c}\frac{j_0}{\sqrt{g_{00}}}\,, \qquad j^i = h^i_\alpha j^\alpha$$

of the four-dimensional current vector j^α of the electromagnetic field (also known as the shift current).

The first equation of Group I is the Biot-Savart law, the second is Gauss' theorem, both in chr.inv.-notation. The first and second equations of Group II represent a chr.inv.-notation of Faraday's law of electromagnetic induction and the conditions for the absence of magnetic charges, respectively.

In particular, the 1st equation in Group II shows that, if the observer's reference space does not rotate, then ${}^*\nabla_i H^{*i} = 0$ (the magnetic field is homogeneous), while the electric field is not, ${}^*\nabla_i E^i = 4\pi\rho$ (the 1st equation in Group I). Therefore, a "magnetic charge", if it really exists, is directly connected with the rotation of space itself.

The 9th equation of the equation system mentioned above is Lorentz' condition

$$\nabla_\sigma A^\sigma = 0\,,$$

which is the conservation condition for the four-dimensional electromagnetic field potential A^α. The 10th equation that makes this system definite (the number of equations in this system must be the same as the number of unknowns), is the

*The chr.inv.-Maxwell equations were first deduced in the late 1960s independently by Nikolai Pavlov and José del Prado (unpublished). Zelmanov asked these students to do it as homework. These equations are deduced on the basis of the chr.inv.-projections of the absolute divergence of a 2nd rank antisymmetric tensor (page 19), as well as the chr.inv.-projections of the absolute divergence of its dual pseudotensor (page 20).

law of conservation of electric charge (known also as the continuity equation)

$$\nabla_\sigma j^\sigma = 0\,,$$

which is the mathematical notation of the fact that electric charge cannot be destroyed, but merely redistributed between the charged bodies in contact.

Using the chr.inv.-formula for the divergence of an arbitrary vector field (see page 18), we obtain the Lorentz condition and the continuity condition in chr.inv.-form

$$\frac{1}{c}\frac{{}^*\partial\varphi}{\partial t} + \frac{\varphi}{c}D + {}^*\nabla_i\, q^i - \frac{1}{c^2}F_i q^i = 0\,,$$

$$\frac{{}^*\partial\rho}{\partial t} + \rho D + {}^*\nabla_i\, j^i - \frac{1}{c^2}F_i\, j^i = 0\,,$$

or, replacing the regular chr.inv.-divergence with the physical chr.inv.-divergence (see page 18), we finally have

$$\frac{1}{c}\frac{{}^*\partial\varphi}{\partial t} + \frac{\varphi}{c}D + {}^*\widetilde{\nabla}_i\, q^i = 0\,,$$

$$\frac{{}^*\partial\rho}{\partial t} + \rho D + {}^*\widetilde{\nabla}_i\, j^i = 0\,.$$

With the above chr.inv.-Lorentz condition and the chr.inv.-continuity equation, the mentioned system of 10 equations that completely characterizes any electromagnetic field in the four-dimensional pseudo-Riemannian space is complete.

Now consider the energy-momentum tensor of an electromagnetic field. It has the form

$$T^{\mu\nu} = \frac{1}{4\pi}\left(-F^{\mu\sigma}F^{\nu\cdot}_{\cdot\sigma} + \frac{1}{4}\,g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right).$$

This tensor is symmetric: $T^{\mu\nu} = T^{\nu\mu}$. For this reason, its chr.inv.-projections are calculated as for any symmetric tensor of the 2nd rank (see page 6)

$$q = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c\,T^i_0}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik}$$

and have the following form

$$q = \frac{E^2 + H^{*2}}{8\pi}\,,$$

$$J^i = \frac{c}{4\pi}\,\varepsilon^{ikm}E_k H_{*m}\,,$$

$$U^{ik} = q c^2 h^{ik} - \frac{c^2}{4\pi}\left(E^i E^k + H^{*i}H^{*k}\right),$$

where $E^2 = h_{ik}E^iE^k$ and $H^{*2} = h_{ik}H^{*i}H^{*k}$. These projections have the following physical sense: the scalar q is the physically observable energy density of the electromagnetic field, J^i is the physically observable density of the field momentum (the chr.inv.-Poynting vector), and U^{ik} is the physically observable density of the field momentum flux (the chr.inv.-stress tensor).

Any electrically charged particle travelling in an electromagnetic field deviates from a geodesic trajectory due to the Lorentz force acting on its electric charge e from the electromagnetic field. The Lorentz force in the four-dimensional pseudo-Riemannian space has the form

$$\Phi^\alpha = \frac{e}{c}\,F^{\alpha\cdot}_{\cdot\sigma}U^\sigma, \qquad U^\alpha = \frac{dx^\alpha}{ds}\,,$$

where U^α is the four-dimensional velocity of the charged particle. Respectively, the four-dimensional equations of motion of a charged particle in an electromagnetic field (determined by the electromagnetic field tensor $F_{\alpha\beta}$) have the form

$$\frac{dP^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}P^\mu U^\nu = \frac{e}{c^2}\,F^{\alpha\cdot}_{\cdot\beta}U^\beta,$$

and their chr.inv.-projections

$$\frac{dE}{d\tau} - m\,F_i\mathrm{v}^i + m\,D_{ik}\,\mathrm{v}^i\mathrm{v}^k = -\,e\,E_i\,\mathrm{v}^i,$$

$$\frac{d\,(m\mathrm{v}^i)}{d\tau} - mF^i + 2m\,(D^i_k + A^{\cdot i}_{k\cdot})\,\mathrm{v}^k + m\,\Delta^i_{nk}\,\mathrm{v}^n\mathrm{v}^k = -\,e\left(E^i + \frac{1}{c}\,\varepsilon^{ikm}\,\mathrm{v}_k\,H_{*m}\right)$$

are the chr.inv.-equations of motion of the charged particle. Here, $E = mc^2$ is the relativistic energy of the particle, so the first (scalar) equation is the theorem of live forces represented in chr.inv.-form.

The above chr.inv.-equations of motion show how the observed motion of charged particles is affected by the physically observable gravitational inertial force F^i, rotation A_{ik}, deformation D_{ik} and inhomogeneity Δ^i_{kn} of the observer's reference space.

Zelmanov had also introduced the chr.inv.-curvature tensor. It is deduced similarly to the Riemann-Christoffel tensor from the non-commutativity of the 2nd chr.inv.-derivatives of an arbitrary vector

$${}^*\nabla_i{}^*\nabla_k\, Q_l - {}^*\nabla_k{}^*\nabla_i\, Q_l = \frac{2A_{ik}}{c^2}\frac{{}^*\partial Q_l}{\partial t} + H^{\cdots j}_{lki\cdot}\,Q_j\,,$$

where the 4th rank chr.inv.-tensor

$$H^{\cdots j}_{lki\cdot} = \frac{{}^*\partial\Delta^j_{il}}{\partial x^k} - \frac{{}^*\partial\Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\,\Delta^j_{km} - \Delta^m_{kl}\,\Delta^j_{im}$$

is the basis for the chr.inv.-curvature tensor C_{lkij}, which has all properties of the Riemann-Christoffel tensor in the observer's three-dimensional spatial section, and its contraction gives the observable chr.inv.-scalar curvature C

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right),$$

$$C_{lk} = C^{\cdots i}_{lki\cdot}\,, \qquad C = h^{lk}C_{lk}\,,$$

where

$$H_{lkij} = C_{lkij} + \frac{1}{2}\left(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + A_{kl}D_{ij} + A_{li}D_{jk}\right),$$

$$H_{lk} = C_{lk} + \frac{1}{2}\left(A_{kj}D^{j}_{l} + A_{lj}D^{j}_{k} + A_{kl}D\right),$$

$$H = h^{lk}H_{lk} = C\,.$$

The above formulae show that the observed curvature of a space depends on not only the gravitational inertial force acting in the local reference space of the observer, but also the rotation and deformation of his reference space, and, therefore, does not vanish in the absence of the gravitational field. If the space does not rotate, then we have $H_{lkij} = C_{lkij}$. This is as well true for H_{lk} and C_{lk}. In this particular case, the tensor $C_{lk} = h^{ij}C_{ilkj}$ has the form

$$C_{lk} = \frac{{}^*\partial}{\partial x^k}\left(\frac{{}^*\partial \ln\sqrt{h}}{\partial x^l}\right) - \frac{{}^*\partial \Delta^i_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^i_{km} - \Delta^m_{kl}\frac{{}^*\partial \ln\sqrt{h}}{\partial x^m}\,.$$

Zelmanov had also deduced chr.inv.-projections for the Riemann-Christoffel curvature tensor

$$R^{i}_{\cdot jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^l} + \Gamma^i_{kp}\Gamma^p_{lj} - \Gamma^i_{lp}\Gamma^p_{kj}\,.$$

The Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ is symmetric with respect to transposition over a pair of its indices and antisymmetric within each pair of the indices. Therefore, it has three chr.inv.-projections as follows

$$X^{ik} = -c^2\,\frac{R^{\cdot i \cdot k}_{0 \cdot 0 \cdot}}{g_{00}}\,, \qquad Y^{ijk} = -c\,\frac{R^{\cdot ijk}_{0\cdots}}{\sqrt{g_{00}}}\,, \qquad Z^{ijkl} = c^2 R^{ijkl}.$$

Substituting the necessary components of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ into these formulae and then lowering the indices, Zelmanov had obtained the chr.inv.-projections of the Riemann-Christoffel tensor in the form

$$X_{ij} = \frac{{}^*\partial D_{ij}}{\partial t} - \left(D^l_i + A^{\cdot l}_{i\cdot}\right)\left(D_{jl} + A_{jl}\right) + \left({}^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2}F_i F_j\,,$$

$$Y_{ijk} = {}^*\nabla_i\left(D_{jk} + A_{jk}\right) - {}^*\nabla_j\left(D_{ik} + A_{ik}\right) + \frac{2}{c^2}A_{ij}F_k\,,$$

$$Z_{iklj} = D_{ik}D_{lj} - D_{il}D_{kj} + A_{ik}A_{lj} - A_{il}A_{kj} + 2A_{ij}A_{kl} - c^2 C_{iklj}\,,$$

where we have $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$, as in the Riemann-Christoffel tensor. Contraction of the observable spatial projection Z_{iklj} step-by-step as $Z_{il} = h^{kj}Z_{iklj}$ and $Z = h^{il}Z_{il}$ gives

$$Z_{il} = D_{ik}D^k_l - D_{il}D + A_{ik}A^{\cdot k}_{l\cdot} + 2A_{ik}A^{k\cdot}_{\cdot l} - c^2 C_{il}\,,$$

$$Z = h^{il}Z_{il} = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2 C\,.$$

Using the above, Zelmanov was able to deduce chr.inv.-projections for Einstein's field equations

$$R_{\alpha\beta} - \frac{1}{2}\,g_{\alpha\beta}R = -\varkappa\,T_{\alpha\beta} + \lambda\,g_{\alpha\beta}\,,$$

where he used $\varkappa = \frac{8\pi G}{c^2}$ instead of $\varkappa = \frac{8\pi G}{c^4}$ as used by Landau and Lifshitz in their *The Classical Theory of Fields* [8]. To understand the reason, consider the chr.inv.-projections of the energy-momentum tensor $T_{\alpha\beta}$ of a distributed matter, which are calculated according to the rule

$$\varrho = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c\,T^i_0}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik}$$

as for any symmetric tensor of the 2nd rank (see page 6). The scalar ϱ is the physically observable mass density of the distributed matter, J^i is its physically observable momentum density, and U^{ik} is its physically observable momentum flux density (stress-tensor). Ricci's tensor $R_{\alpha\beta}$ has the dimension [cm^{-2}]. This means that the scalar chr.inv.-projection of the field equations, $\frac{G_{00}}{g_{00}} = -\frac{\varkappa T_{00}}{g_{00}} + \lambda$, as well as $\frac{\varkappa T_{00}}{g_{00}} = \frac{8\pi G\varrho}{c^2}$ have the same dimension [cm^{-2}]. Hence, the energy-momentum tensor has the dimension of mass density [gram/cm^3]. Therefore, if we used $\varkappa = \frac{8\pi G}{c^4}$ on the right hand side of the field equations, then we would not use the energy-momentum tensor $T_{\alpha\beta}$ itself, but $c^2 T_{\alpha\beta}$ as Landau and Lifshitz did.

Taking all the above into account, Zelmanov had obtained the chr.inv.-projections of Einstein's field equations. They are called the chr.inv.-Einstein equations and have the form

$$\frac{{}^*\partial D}{\partial t} + D_{jl}D^{jl} + A_{jl}A^{lj} + {}^*\nabla_j F^j - \frac{1}{c^2}F_j F^j = -\frac{\varkappa}{2}\left(\varrho c^2 + U\right) + \lambda c^2,$$

$${}^*\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) + \frac{2}{c^2}F_j A^{ij} = \varkappa\,J^i,$$

$$\frac{{}^*\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D^j_k + A^{\cdot j}_{k\cdot}\right) + DD_{ik} + 3A_{ij}A^{\cdot j}_{k\cdot} - \frac{1}{c^2}F_i F_k + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - c^2 C_{ik} = \frac{\varkappa}{2}\left(\varrho\,c^2 h_{ik} + 2\,U_{ik} - U h_{ik}\right) + \lambda\,c^2 h_{ik}\,.$$

In addition, the energy-momentum tensor $T_{\alpha\beta}$ of the distributed matter must satisfy the conservation law

$$\nabla_\sigma T^{\sigma\alpha} = 0\,,$$

the chr.inv.-projections of which are calculated as for the absolute divergence of any symmetric tensor of the 2nd rank (see page 20), and are chr.inv.-conservation law equations

$$\frac{{}^*\partial\varrho}{\partial t} + D\varrho + \frac{1}{c^2}D_{ij}U^{ij} + {}^*\widetilde{\nabla}_i J^i - \frac{1}{c^2}F_i J^i = 0\,,$$

$$\frac{{}^*\partial J^k}{\partial t} + DJ^k + 2\left(D^k_i + A^{\cdot k}_{i\cdot}\right)J^i + {}^*\widetilde{\nabla}_i U^{ik} - \varrho F^k = 0\,.$$

So, we have presented here Zelmanov's mathematical apparatus of chronometric invariants, which are physical observables in General Relativity. This mathematical apparatus is given here in its entirety and in the form it was introduced by Zelmanov in 1944 (except for the chr.inv.-Maxwell equations, the chr.inv.-d'Alembert and chr.inv.-Laplace operators, which were deduced later). The above description of this mathematical apparatus contains all its foundations and definitions, tensor calculus in terms of chronometric invariants, as well as the most common equations used in General Relativity, which are also expressed in terms of chronometric invariants. All this is collected here in one article, which is very convenient. Even if we have missed some details, these details are not essential for understanding and working with this mathematical apparatus.

Zelmanov's mathematical apparatus was applied to many problems of General Relativity. In general, Zelmanov always said that he liked creating "mathematical tools" more than applying them. Nevertheless, his contribution to relativistic cosmology, as well as his calculation of the main effects of General Gelativity and the basics of electrodynamics in terms of chronometric invariants, are significant. We also made a contribution: the list of our works, published in English and French, can be found just after the References*.

We recommend the present article to all those readers who would like to work independently in the field of General Relativity using the mathematical apparatus of chronometric invariants. Good luck!

Submitted on January 3, 2023

References

1. Zelmanov A. L. Chronometric Invariants. Translated from the 1944 PhD thesis, American Research Press, Rehoboth (New Mexico), 2006.
2. Zelmanov A. L. Chronometric invariants and accompanying frames of reference in the General Theory of Relativity. *Soviet Physics Doklady*, 1956, v. 1, 227–230 (translated from *Doklady Academii Nauk SSSR*, 1956, v. 107, no. 6, 815–818).
3. Zelmanov A. L. On the relativistic theory of an anisotropic inhomogeneous universe. *The Abraham Zelmanov Journal*, 2008, v. 1, 33–63 (translated from the thesis of the *6th Soviet Conference on the Problems of Cosmogony*, held in 1957 in Moscow, SSSR Acad. Science Publishers, Moscow, 1959, 144–174).
4. Rabounski D and Borissova L. Particles Here and Beyond the Mirror. The 3rd expanded edition, American Research Press, Rehoboth, New Mexico, 2012 (first published in 1999, in Russian; the 1st English edition — in 2001). Rabounski D. et Borissova L. Particules de l'univers et au delà du miroir. American Research Press, Rehoboth, New Mexico, 2012 (French translation).
5. Borissova L. and Rabounski D. Fields, Vacuum, and the Mirror Universe. The 2nd expanded edition, Svenska fysikarkivet, Stockholm, 2009 (first published in 1999, in Russian; the 1st English edition — in 2001). Rabounski D. et Borissova L. Champs, vide, et univers miroir. American Research Press, Rehoboth, New Mexico, 2010 (French translation).
6. Borissova L. and Rabounski D. Inside Stars: A Theory of the Internal Constitution of Stars, and the Sources of Stellar Energy According to General Relativity. 2nd expanded edition, American Research Press, Rehoboth, New Mexico, 2014.
7. Rabounski D. and Borissova L. Non-quantum teleportation in a rotating space with a strong electromagnetic field. *Progress in Physics*, 2022, no. 1, 31–49.
8. Landau L. D. and Lifshitz E. M. The Classical Theory of Fields. Pergamon Press, Oxford, 1951 (translated from the 1st Russian edition published in 1939).
9. Cattaneo C. General Relativity: relative standard mass, momentum, energy, and gravitational field in a general system of reference. *Nuovo Cimento*, 1958, v. 10, 318–337.
10. Cattaneo C. On the energy equation for a gravitating test particle. *Nuovo Cimento*, 1959, v. 11, 733–735.
11. Cattaneo C. Conservation laws in General Relativity. *Nuovo Cimento*, 1959, v. 13, 237–240. vol. 11, 733–735.
12. Cattaneo C. Problèmes d'interprétation en Relativité Générale. *Colloques Internationaux du Centre National de la Recherche Scientifique*, no. 170 "Fluides et champ gravitationel en Relativité Générale", Éditions du Centre National de la Recherche Scientifique, Paris, 1969, 227–235.
13. Hafele J. Performance and results of portable clocks in aircraft. PTTI 3rd Annual Meeting, November 16–18, 1971, 261–288.
14. Hafele J. and Keating R. Around the world atomic clocks: predicted relativistic time gains. *Science*, July 14, 1972, v. 177, 166–168.
15. Hafele J. and Keating R. Around the world atomic clocks: observed relativistic time gains. *Science*, July 14, 1972, v. 177, 168–170.
16. Demonstrating relativity by flying atomic clocks. *Metromnia*, the UK's National Measurement Laboratory Newsletter, issue 18, Spring 2005.
17. Raschewski P. K. Riemannsche Geometrie und Tensoranalysis. Deutscher Verlag der Wissenschaften, Berlin, 1959; reprinted by Verlag Harri Deutsch, Frankfurt am Main, 1993.
18. Schouten J. A. und Struik D. J. Einführung in die neuren Methoden der Differentialgeometrie. Noordhoff, Groningen, 1938 (first published in *Zentralblatt für Mathematik*, 1935, Bd. 11 und Bd. 19).

Applying Chronometric Invariants

1. Borissova L. B. (née Grigor'eva) Chronometrically invariant representation of the classification of the Petrov gravitational fields. *Soviet Physics Doklady*, 1970, v. 15, 579–582 (translated from *Doklady Acad. Nauk SSSR*, 1970, v. 192, no. 6, 1251–1254).
2. Borissova L. B. (née Grigor'eva) and Zakharov V. D. Spaces of recurrent curvature in the General Theory of Relativity. *Soviet Physics Doklady*, 1973, v. 17, 1160–1163 (translated from *Doklady Acad. Nauk SSSR*, 1972, v. 207, no. 4, 814–816).

*It is necessary to say a word about the authorship of those articles in this list, which were published before 1991. It was the dark time of the communist dictatorship, when the personal contribution of a researcher, especially a woman, was neglected. Therefore, when L. Borisova submitted an article for publication through her superiors (because there was no other way to submit at that time), she could often find their names added to the submission. She was allowed to publish her articles only under her own name only after great troubles and a scandal. As one of the superiors publicly stated: "Science is a man's business. We will not allow this 'Einstein in a skirt' to be present in science." Even Zelmanov, who took custody of her from her student years, could not do anything against this suppression and lawlessness. As a result, those persons whose names can be found as "co-authors" in some of her publications before 1991 had nothing to do with her research: they, having an administrative power, simply added their names to her submissions. Their names must therefore be excluded from those published articles and forgotten (despite the fact that we mentioned them in the bibliography for this article). Fortunately, this dark era of our lives ended in 1991 after the collapse of the USSR and everything connected with it. All the mathematical problems that we considered in our works (from our student years to the present day) were posed and solved only by us, individually or together, but without any assistance or advice of a "supervisor" or another person.

3. Borissova L. B. and Zakharov V. D. Gravitoinertial waves in vacuum. *Russian Physics Journal*, 1974, v. 17, no. 12, 1723–1728 (translated from *Izvestiia Vysshikh Uchebnykh Zavedenii*, *Fizika*, 1974, no. 12, 106–113).
4. Borissova L. B. Relative oscillations of test particles in comoving reference frames. *Soviet Physics Doklady*, 1976, v. 20, 816–819 (translated from *Doklady Acad. Nauk SSSR*, 1975, v. 225, no. 4, 786–789).
5. Borissova L. B. and Zakharov V. D. A system of probe particles in the field of planar gravitational waves. Part I. *Russian Physics Journal*, 1976, v. 19, no. 12, 1617–1619 (translated from *Izvestiia Vysshikh Uchebnykh Zavedenii*, *Fizika*, 1976, no. 12, 108–111).
6. Borissova L. B. and Zakharov V. D. A system of probe particles in the field of planar gravitational waves. Part II. *Russian Physics Journal*, 1976, v. 19, no. 12, 1620–1624 (translated from *Izvestiia Vysshikh Uchebnykh Zavedenii*, *Fizika*, 1976, no. 12, 111–117).
7. Borissova L. B. Quadrupole mass-detector in field of weak plane gravitational fields. *Russian Physics Journal*, 1978, v. 21, no. 10, 1341–1344 (translated from *Izvestiia Vysshikh Uchebnykh Zavedenii*, *Fizika*, 1978, no. 10, 109–114).
8. Borisova L.B. and Melnikov V.N. Laser interferometer in variable gravitational field. *Measurement Techniques*, 1985, v. 28, no. 4, 301–307 (translated from *Izmeritel'naya Tekhnika*, 1985, no. 4, 16–19).
9. Borisova L. B. and Melnikov V. N. Relativistic corrections to readings from a portable clock. *Measurement Techniques*, 1988, v. 31, no. 4, 323–327 (translated from *Izmeritel'naya Tekhnika*, 1988, no. 4, 13–15).
10. Borisova L. B., Bronnikov K. A., Melnikov V. N. Taking into account gravitational and relativistic effects in maintaining a unified time scale on the Earth and surrounding space. *Measurement Techniques*, 1988, v. 31, no. 5, 450–455 (translated from *Izmeritel'naya Tekhnika*, 1988, no. 5, 31–33).
11. Rabounski D. and Borissova L. Particles Here and Beyond the Mirror. Editorial URSS Publishers, Moscow, 2001.
12. Borissova L. and Rabounski D. Fields, Vacuum, and the Mirror Universe. Editorial URSS Publishers, Moscow, 2001.
13. Rabounski D. A new method to measure the speed of gravitation. *Progress in Physics*, 2005, no. 1, 3–6.
14. Borissova L. and Rabounski D. On the possibility of instant displacements in the space-time of General Relativity. *Progress in Physics*, 2005, no. 1, 17–19.
15. Rabounski D. A theory of gravity like electrodynamics. *Progress in Physics*, 2005, no. 2, 15–29.
16. Borissova L. Gravitational waves and gravitational inertial waves in the General Theory of Relativity: A theory and experiments. *Progress in Physics*, 2005, no. 2, 30–62.
17. Rabounski D., Borissova L., Smarandache F. Entangled states and quantum causality threshold in the General Theory of Relativity. *Progress in Physics*, 2005, no. 2, 101–107.
18. Rabounski D., Smarandache F., Borissova L. Neutrosophic Methods in General Relativity. Hexis, Phoenix, Arizona, 2005.
19. Rabounski D. Zelmanov's Anthropic Principle and the Infinite Relativity Principle. *Progress in Physics*, 2006, no. 1, 35–37.
20. Rabounski D. Correct linearization of Einstein's equations. *Progress in Physics*, 2006, no. 2, 3–5.
21. Rabounski D. and Borissova L. Exact theory of a gravitational wave detector. New experiments proposed. *Progress in Physics*, 2006, no. 2, 31–38.
22. Rabounski D., Smarandache F., Borissova L. S-denying of the signature conditions expands General Relativity's space. *Progress in Physics*, 2006, no. 3, 13–19.
23. Borissova L. and Smarandache F. Positive, neutral and negative mass-charges in General Relativity. *Progress in Physics*, 2006, no. 3, 51–54.
24. Rabounski D. New effect of General Relativity: Thomson dispersion of light in stars as a machine producing stellar energy. *Progress in Physics*, 2006, no. 4, 3–10.
25. Borissova L. Preferred spatial directions in the Universe: a General Relativity approach. *Progress in Physics*, 2006, no. 4, 51–58.
26. Borissova L. Preferred spatial directions in the Universe. Part II. Matter distributed along orbital trajectories, and energy produced from it. *Progress in Physics*, 2006, no. 4, 59–64.
27. Rabounski D. The theory of vortical gravitational fields. *Progress in Physics*, 2007, no. 2, 3–10.
28. Borissova L. Forces of space non-holonomity as the necessary condition for motion of space bodies. *Progress in Physics*, 2007, no. 2, 11–16.
29. Rabounski D. and Borissova L. A theory of the Podkletnov effect based on General Relativity: Anti-gravity force due to the perturbed non-holonomic background of space. *Progress in Physics*, 2007, no. 3, 57–80.
30. Borissova L. and Rabounski D. On the nature of the microwave background at the Lagrange 2 point. *Progress in Physics*, 2007, no. 4, 84–95.
31. Rabounski D and Borissova L. Particles Here and Beyond the Mirror. The 2nd expanded edition, Svenska fysikarkivet, Stockholm, 2008.
32. Borissova L. and Rabounski D. PLANCK, the satellite: A new experimental test of General Relativity. *Progress in Physics*, 2008, no. 2, 3–14.
33. Rabounski D. An explantion of Hubble redshift due to the global non-holonomity of space. *Progress in Physics*, 2009, no. 1, L1–L2.
34. Rabounski D. Hubble redshift due to the global non-holonomity of space. *The Abraham Zelmanov Journal*, 2009, v. 2, 11–28.
35. Rabounski D. On the speed of rotation of the isotropic space: Insight into the redshift problem. *The Abraham Zelmanov Journal*, 2009, v. 2, 208–223.
36. Borissova L. The gravitational field of a condensed matter model of the Sun: The space breaking meets the Asteroid strip. *The Abraham Zelmanov Journal*, 2009, v. 2, 224–260.
37. Borissova L. and Rabounski D. Fields, Vacuum, and the Mirror Universe. The 2nd expanded edition, Svenska fysikarkivet, Stockholm, 2009.
38. Rabounski D. et Borissova L. Particules de l'Univers et au delà du Miroir. American Research Press, Rehoboth, New Mexico, 2012 (French translation).
39. Borissova L. The Solar System according to General Relativity: The Sun's space breaking meets the Asteroid strip. *Progress in Physics*, 2010, no. 2, 43–47.
40. Borissova L. De Sitter bubble as a model of the observable Universe. *The Abraham Zelmanov Journal*, 2010, v. 3, 3–24.
41. Borissova L. Gravitational waves and gravitational inertial waves according to the General Theory of Relativity. *The Abraham Zelmanov Journal*, 2010, v. 3, 25–70.
42. Rabounski D. and Borissova L. A theory of frozen light according to General Relativity. *The Abraham Zelmanov Journal*, 2011, v. 4, 3–27.
43. Rabounski D. Cosmological mass-defect — a new effect of General Relativity. *The Abraham Zelmanov Journal*, 2011, v. 4, 137–161.
44. Rabounski D. Non-linear cosmological redshift: The exact theory according to General Relativity. *The Abraham Zelmanov Journal*, 2012, v. 5, 3–30.
45. Rabounski D. On the exact solution explaining the accelerate expanding Universe according to General Relativity. *Progress in Physics*, 2012, no. 2, L1–L6.
46. Rabounski D and Borissova L. Particles Here and Beyond the Mirror. The 3rd expanded edition, American Research Press, Rehoboth, New Mexico, 2012.

47. Rabounski D. et Borissova L. Particules de l'Univers et au delà du Miroir. American Research Press, Rehoboth, New Mexico, 2012 (French translation).
48. Borissova L. and Rabounski D. Inside Stars: A Theory of the Internal Constitution of Stars, and the Sources of Stellar Energy According to General Relativity. American Research Press, Rehoboth, New Mexico, 2013.
49. Borissova L. and Rabounski D. Inside Stars: A Theory of the Internal Constitution of Stars, and the Sources of Stellar Energy According to General Relativity. 2nd expanded edition, American Research Press, Rehoboth, New Mexico, 2014.
50. Borissova L. A telemetric multispace formulation of Riemannian geometry, General Relativity, and cosmology: Implications for relativistic cosmology and the true reality of time. *Progress in Physics*, 2017, no. 2, 57–75.
51. Borissova L. and Rabounski D. Cosmological redshift in the de Sitter stationary Universe. *Progress in Physics*, 2018, no. 1, 27–29.
52. Rabounski D. and Borissova L. Non-quantum teleportation in a rotating space with a strong electromagnetic field. *Progress in Physics*, 2022, no. 1, 31–49.
53. Rabounski D. and Borissova L. Deflection of light rays and mass-bearing particles in the field of a rotating body. *Progress in Physics*, 2022, no. 1, 50–55.
54. Rabounski D. and Borissova L. Length stretching and time dilation in the field of a rotating body. *Progress in Physics*, 2022, no. 1, 62–65.